Groups And Burnside Problem

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This project is focused on the combinatorial-geometrical method that has enabled to solve several complicated problems within the group theory like the construction of a infinite finitely generated group with identity $x^n = 1$ to name one (The restricted Burnside problem). The goal of this project is the construction of this group. The following results were obtained by A. Yu. Olshanskii. The fact that group relations' consequences could be presented geometrically is the essence of the method.

We should introduce some useful ideas and definitions. Then we begin constructing diagrams on a plane. Usually the diagram is a polygon map. So it is interesting to handle the following preliminary problems.

◆ **A1.** *A convex 1993-gon was cut to convex 7-gons. Prove that exist four neighboring vertices of the 1993-gon which belong to same 7-gon. (A vertex of a 7-gon cannot belong to an edge of the 1993-gon.)*

◆ **A2.** *Can one cut a plane: a) to convex 7-gons? b) to equal convex 7-gons?*

◆ **A3.** *Can one cut a plane to convex 7-gons such that any unite circle intersects with less then million of them?*

◆ **A4.** *Let us consider a plane divided into 7-gons which diameters are less or equal to 1. Fix a point O. Let $N(R)$ be the number of 7-gons falling into the circle with diameter R and center O. Prove that there exists $\lambda > 1$ such that $N(R) > \lambda^R$.*

Let us consider an alphabet $\boldsymbol{L}$. Let $\boldsymbol{L}$ contain the following letters:

$$\boldsymbol{a},\ \boldsymbol{a^{-1}},\ \boldsymbol{b},\ \boldsymbol{b^{-1}},\ \boldsymbol{c},\ \boldsymbol{c^{-1}},\ \dots$$

For any letter in our alphabet there exists a pair: $\boldsymbol{a}$ and $\boldsymbol{a^{-1}}$, $\boldsymbol{b}$ and $\boldsymbol{b^{-1}}$ and so on. These letters are called *inverse* letters. These letters can be used to construct words (some finite sequences of letters). Examples: $\boldsymbol{aba}$, $\boldsymbol{aba^{-1}ab^{-1}c}$, $\boldsymbol{a^{-1}a^{-1}bbca^{-1}}$. Words can be transformed by either cancelling or inserting two neighboring inverse letters. Thus $\boldsymbol{aba^{-1}ab^{-1}c}$ transforms into $\boldsymbol{ac}$: $\boldsymbol{ab\underline{a^{-1}a}b^{-1}c} \equiv \boldsymbol{a\underline{bb^{-1}}c} \equiv \boldsymbol{ac}$. Some words (for example, $\boldsymbol{b^{-1}aa^{-1}b}$) can be transformed into an empty word. Let this word be labeled zero.

Suppose that some words equal to $\boldsymbol{1}$. For example, $\{\boldsymbol{aba^{-1}b^{-1}} = \boldsymbol{1};\ \boldsymbol{ca} = \boldsymbol{1}\}$. Such relations are said to be *defining relations*. We can use these equalities to obtain new ones. For instance, suppose that $\boldsymbol{aba^{-1}b^{-1}} = \boldsymbol{1}$. Let us add a word $\boldsymbol{ba}$ to both sides on the right. ("multiply by $\boldsymbol{ba}$ on the right"), We obtain $\boldsymbol{aba^{-1}b^{-1}ba} = \boldsymbol{ba}$. Having cancelled inverse letters on the left side, we obtain $\boldsymbol{ab} = \boldsymbol{ba}$. Having added $\boldsymbol{c}$ to both sides of the equality (on the left) we get $\boldsymbol{cab} = \boldsymbol{cba}$. Since $\boldsymbol{ca} = \boldsymbol{1}$, we get $\boldsymbol{b} = \boldsymbol{cba}$.

After considering the defining relations we see that some words are *equivalent* or *equal*. We can multiply an equality by the same factor or cancel inverse letters. Thus we obtain new relations (equal words). Suppose we have some defining relations. Let us consider two words.

Can we find out whether these words are equal? The thing is that there is no general algorithm. This is a very important result in higher algebra. Nevertheless if the defining relations satisfy certain terms, such an algorithm exists in fact.

The main issue. Let the common begining of any pair of defining relations $A = 1$ and $B = 1$ be either empty or its length constitutes less then $^1/_6$ of the A and B lenght. Thus there exists an algorithm enabling for any pair of words to find out if they are equal or not.

The main issue will be discussed below. First we should practice with simple examples. Herewith, we assume that the alphabet consists of the letters from the defining relations only.

Let us consider an example. Suppose $aba = 1$, $bab = 1$. Let us prove that $a = b$. Indeed, $a \equiv abaa^{-1}b^{-1} \equiv a^{-1}b^{-1} \equiv a^{-1}b^{-1}bab \equiv b$. So there exist three nonequivalent words: (1, a, a^2).

Consequently, if a letter occurs in a word twice, we shall use powers in our notation. So we shall write a^k instead of $\underbrace{aaa\ldots aa}_{k}$. If a power is negative, we put $x^k = (x^{-1})^{-k}$. In particular, $a^{-3} = (a^{-1})^3$.

♦ **B1.** *Let $ba = ab^k$ be a defining relation such that k is a nonzero integer. Prove that any word can be transformed to the form $a^m b^n$; here m, n are integers.*

♦ **B2.** *Consider defining relations $a^4 = b^3 = (ab)^2 = 1$. How many different words there are?*

♦ **B3.** *Consider defining relations $aba^{-2}ba = b^3 = 1$. How many different words there are?*

♦ **B4.** *Consider defining relations $a^2 = b^2 = (ab)^n = 1$. How many different words there are?*

There exists a way which lets illustrate consequences from defining relations. Look at the pictures illustrating the two following consequences: $a^2 b^3 = b^3 a^2$ with the relation $ab = ba$; and $b^6 = 1$ with the relations $b^2 = a$ and $a^3 = 1$. If we move around any cell then we read one of the defining words, and if we move around the whole map then we read the consequence. If we move in the opposite direction then we read letters as inverse ones.
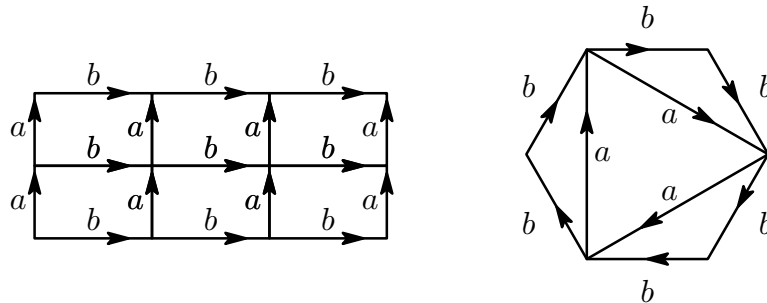

рис. 1   рис. 2

Hence the plane can be divided into some polygons. Now we shall give some definitions. The area inside any polygon is said to be *a cell*. The e dges of the polygons are called *the edges*. Let us write a letter near each edge on the map such that the words written arround any cell correspond to the defining relations words.

♦ **B5.** *Draw a map for the consequence* $a^2b^2c^2 = 1$ *using the defining relations* $a^3 = 1$, $b^3 = 1$, $c^3 = 1$, $abc = 1$.

♦ **B6.** *Draw a map for the consequence* $ab^{-1}aba^{-1}b = 1$ *using the defining relations* $a^3 = 1$, $b^3 = 1$, $abab = 1$.

In fact, the discussed above structures of words with relations have their own name.

Let us consider finite or infinite set of elements G. We assume the law, according to which a third element z is uniquely obtained from any two equal or different elements x and y of such a set, is known. This operation is called *composition* or *symbolic multiplication* of elements and it's result is called the *product* of the elements x and y. We denote the product of x and y by $x * y$. We note that the law of composition may be such that the result depends on the order of the elements multiplied. So $x * y$ is not in general equal to $y * x$.

We will discuss the operations such that an equality $(x * y) * z = x * (y * z)$ holds for any x, y, z from G. In these cases we say that the operation $*$ is *associative*.

Let G be a set of all words in our finite alphabet. We can assign a product operation on G: a word W is called the product of two words A and B if W is a result of attaching of B to A. Specifically, If $A = a_1a_2 \ldots a_k$, $B = b_1b_2 \ldots b_n$ then $W = a_1a_2 \ldots a_kb_1b_2 \ldots b_n$. Obviously, this operation is associative. It is easy to understand the meaning of the associativity. The overall result of a product $x_1 * x_2 * \cdots * x_n$ is not depends on brackets placement. So the products $(x_1 * x_2) * (x_3 * x_4)$ and $x_1 * ((x_2 * x_3) * x_4)$ are equal.

Definition. A set G with an operation $*$ is called a *group* if the following three conditions hold:

(i) For any $x, y, z \in G$ the following condition holds: $(x * y) * z = x * (y * z)$. So the operation is assotiative;

(ii) There exists an element 1 in G such that $x * 1 = 1 * x = x$ for any $x \in G$. This element is called an *identity*;

(iii) For any $x \in G$ there exists a $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = 1$.

We shall often ommit the $*$ sign and write (for example) $ab = c$ or $xy = yx$.

The set of all words in a finite alphabet is a group: an empty word is an identity. If we replace any letter in a word with inverse letter and rewrite the word in the reverse order then we obtain an inverse word. For example, the word $z^{-1}y^{-1}x^{-1}d^{-1}c^{-1}b^{-1}a^{-1}$ is inverse for the word $abcdxyz$. And $aba^{-1}c^{-1}ba^{-1}$ is inverse for $ab^{-1}cab^{-1}a^1$. If we write two inverse words one after another then we can obtain an empty word (identity) by several transformations.

Now assume defining relations. It is easy to see that some words become equivalent. Let us consider these words as the same element of the group. So if we want to make some operations with this element we can choose any of these words.

The number of different elements of a group may be finite, in which case the group is called *finite* and the number of its elements is called its *order*. The order of a group is denoted by $|G|$.

A group is abelian (or commutative) if $xy = yx$ for all $x, y \in G$.

♦ **B7.** *Prove that if for any x from a group $x^2 = 1$, then the group is abelian.*

Definition. A nonempty subset $H \subset G$ is called a *subgroup* of a group G if the following conditions hold:

(i) if $a, b \in H$ then $ab \in H$;

(ii) if $a \in H$ then $a^{-1} \in H$.

It is easy to see that the identity of G belongs to H (suppose $a \in H$, then $a^{-1} \in H$ and $aa^{-1} \in H$). Hence H is a group with respect to the mother-group G operation.

Note that there are two trivial subgroups in any given group: the whole group and the group with one element $\{1\}$. These two groups are called *improper*. Any other group is called *proper*. How can one find any proper group? Suppose $a \in G$. Let us find some powers of a. The set $\{a^k\}$ is an abelian (commutative) subgroup in G. This group is called a *cyclic subgroup* of G and is denoted by $\langle a \rangle$. The element a is said to be a *generator* of $\langle a \rangle$. If $\langle a \rangle = G$, then we say that G is a *cyclic group*.

Suppose n is the minimal integer such that $a^n = 1$. The integer n is called an *order* of element a. If there are no such n, then we say that a has an infinite order.

A cyclic group is generated by its unique element. Let us try to generate a group by several elements.

Suppose that the subset $\langle S \rangle$ in G possesses all finite products

$$g_1{}^{\alpha_1} g_2{}^{\alpha_2} \cdots g_k{}^{\alpha_k},$$

$g_i \in S$, $\alpha_i = \pm 1$, $i = 1, \ldots, k$. It is easy to see that $\langle S \rangle$ is a subgroup in G. The inverse element for the product above is

$$g_k{}^{\beta_k} g_{k-1}{}^{\beta_{k-1}} \cdots g_1{}^{\beta_1},$$

$\beta_i = -\alpha_i$. If every element of a group G is the product of a finite number of elements and inverses of elements from S, then we call the subset S a *set of generators* of G and call the elements of S *generating elements*.

Up to this point we considered every word as a finite letter sequence. Nevertheless, let us have a diagram for the defining relation $a_1 a_2 \ldots a_n = 1$. If we look at this diagram, than we cannot find out which letter is the first in the word. Hence it is good to consider a set of all cyclic shifts of a word. This set is called a *cyclic word*. Let a *subset of a cyclic word* (that is not cyclic!) be a subset of one of a word's cyclic shifts. For example, a^2b, aba, ba^2, a^3 are subwords of lenght 3 of the cyclic word a^2ba. A word is called *cyclic irredundant* if every of it's cyclic shifts is irredundant.

Cyclic shifts appear in groups too. Two elements a and b are called *conjuagate* if there exists $x \in G$ such that $a = xbx^{-1}$. It is easy to see that all cyclic shifts are pairwise conjuagate.

♦ **B8.** *Consider a group with some defining relations. Let us take a word. Prove that there exists a conjuagate cyclic irredundant one.*

♦ **B9.** *Пусть группа задана определяющими соотношениями $U_1 = 1$, ..., $U_k = 1$. Докажите, что если $W \equiv 1$ (слово W приводится к пустому), то существуют такие слова $X_1, \ldots X_k$, что слово*

$$X_1 U_1 X_1^{-1} X_2 U_2 X_2^{-1} \ldots X_k U_k X_k^{-1}$$

приводится к W только сокращениями рядом стоящих взаимно-обратных элементов.

♦ **B9.** *Consider the group with the following defining relations: $U_1 = 1, U_2 = 1, \ldots U_k = 1$. Suppose that $W \equiv 1$. Prove that there exist words $X_1, X_2, \ldots X_k$, such that one can transform the word*

$$X_1 U_1 X_1^{-1} X_2 U_2 X_2^{-1} \ldots X_k U_k X_k^{-1}$$

into an empty word using only the cancellations of neighboring inverse letters.

Suppose a group satisfies the following equalities: $a^3 = 1$, $bab^{-1} = c$. Hence we obtain the relation $c^3 = 1$. This conclusion can be illustrated by the picture 3. Indeed, if we move around any triangle cell, then we read the word a^3. If we move around any quadrangle cell, then we read the word $cba^{-1}b^{-1}$. If we move around the edges of the whole map, we read the word c^3. This word is a consequence of the equalities $a^3 = 1$ and $cba^{-1}b^{-1} = 1$
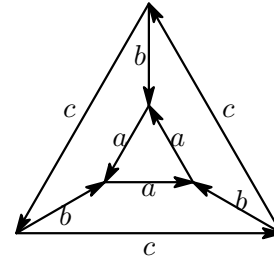

рис. 3

Let us describe how to construct such examples. In order to do this, we consider the conclusion $a^3 = 1, b^2 = a \rightarrow a^3 b^{-1} a^2 b^3 = 1$ in detail. First we convert $a^3 b^{-1} a^2 b^3 = 1$ into the form $(a^3)(b^{-1} a^3 b)(b^{-1} a^{-1} b^2 b)$. (Here we present our word as a product of some defining relation words and its' inverses.) Let us draw a petal for each product efficient. (see pict. 4.)
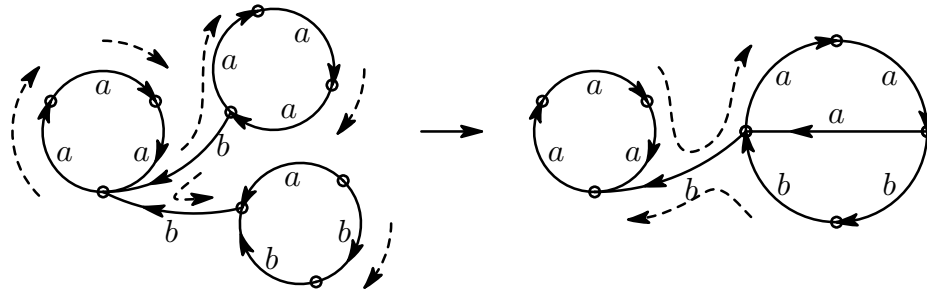

рис. 4

Every petal is a circle with a stem. Then we mark every circle with a letters. Finally we obtain defining relation words (in this example a^3 или $a^{-1} b^2$) on the circles and conjugating words (in this example b^{-1} or an empty word) on the stems. Now walk around all the petals. It is easy to see that we can read the right part of the equality

$$a^3 b^{-1} a^2 b^3 = (a^3)(b^{-1} a^3 b)(b^{-1} a^{-1} b^2 b).$$

We want to obtain the word graphically equal to the left part of this equality. So we need to do some cancellations. We do these cancelations by a conglutination of the stems and two edges of the second and the third circles. Finally we obtain the diagram with the word $a^3 b^{-1} a^2 b^3$ on the outline.

The construction of any other conclusion is performed in a similar way. However there are many different diagrams for one conclusion $W = 1$.

♦ **C1. THE VAN KAMPEN LEMMA.** *Suppose that W is a nonempty word in the alphabet $\bar{L} = L \cup L^{-1} \cup 1$; then $W = 1$ iff there exists a diagram which label graphically equals W.*

Consider the group with the following defining relations: $U = \{U_1 = 1, \ldots, U_k = 1\}$. Let us agree that every word U_i is a cyclic irredundant one and if a word R belongs to the system of the defining relations then R^{-1} also does. Also we may assume that if $XY = 1$ is a defining relation, then $YX = 1$ is a defining relation too. A system of defining relations is *symmetric* if it satisfies all these conditions. It is easy to see that an addition of inverse letters and cycle shifts does not change the set of all consequences. Hence it does not change the group G.

If X is a common beginning of two different words XY_1 и XY_2 from U, then we say that X is a *piece* with respect to U. Consider the group G with the system of defining relations U. Suppose that the lenght of every piece X is less than $\frac{1}{6}$ of the length of any word this piece belongs to. Then we say that the group G is a *group with small cancellations*. This condition means that only small part could be cancelled in the product $U_i U_j$ of defining relations.

If we transform words in a group with small cancellations then we «leave many traces» of the defining relations.

♦ **C2.** *Suppose that G is a group with small cancellations. Let us consider a diagram over G. Suppose that the label of the outline $\phi(q)$ is a cyclic irredundant word and there are no proper subwords in the cyclic word $\phi(q)$ which are equal to 1. Then there exists a cell P with an external arc which length is more than half of the cell perimeter.*

♦ **C3.** *Suppose that G is a group with small cancellations defined by a set of relations. Prove that there exists an algorithm enabling for any pair of words to find out if they are equal or not.*