FUNCTIONAL EQUATIONS

First stage

A) **A functional equation** is an equation including one or more unknown functions (having prescribed domain and range). To solve a functional equation means to find all functions which satisfy it identically. Functional equations arise in various areas of mathematics, usually when we have to describe all functions having given properties.

We start with some typical methods for solving functional equations. **The method of substitution** is often useful. It consists in replacing variables by some new functions (maybe constants) in order to reduce the equation to some more handy form.

1. Solve the following functional equations.

(a)
$$f(x) + 2f\left(\frac{1}{x}\right) = 3x.$$

(b)
$$f(x) + f\left(\frac{x-1}{x}\right) = 2x.$$

(c)
$$f(x+y) + f(x-y) = 2f(x)\cos y.$$

Solution. (a) Denote $y = 1/x$. Then $f\left(\frac{1}{y}\right) + 2f(y) = \frac{3}{y}$ and $f(y) + 2f(\frac{1}{y}) = 3y$. Hence $f(y) = \frac{2}{y} - y$.

(b) Denote $y = \dfrac{x-1}{x}$, then $z = \dfrac{y-1}{y}$. We obtain a system of three linear equations for $f(x), f(y), f(z)$, from which we find

$$f(x) = x + \frac{1}{1-x} - \frac{x-1}{x}.$$

(c) Put $y = \pi/2$, then $f(x + \frac{\pi}{2}) + f(x - \frac{\pi}{2}) = 0$ for any x, hence $f(x+\pi) = -f(x)$. Replace y with $y + \frac{\pi}{2}$, then

$$f(x+y+\frac{\pi}{2}) + f(x-y-\frac{\pi}{2}) = -2f(x)\sin y.$$

Now replace $x - \frac{\pi}{2}$ with x, then

$$f(x+y+\pi) + f(x-y) = -2f(x+\frac{\pi}{2})\sin y,$$

and according to the above

$$f(x+y) - f(x-y) = 2f(x+\frac{\pi}{2})\sin y.$$

For $x = 0$, in view of the original equation we have

$$f(y) = f(0)\cos y + f(\frac{\pi}{2})\sin y.$$

So the desired function must be of the form $f(y) = a\cos y + b\sin y$ where a, b are constants. We easily check that any such function satisfies the original equation.

B) Now we shall consider some specific kinds of functional equations. Many functional equations include some iteration of the unknown function. The following Problem 2 is said to be often given by Feynman to his young colleagues.

2. Does there exist a function $f(x) : R \to R$ such that $f(f(x)) = x^2 - 2$ for all real x?

Solution. No. Suppose $g(x) := f(f(x)) = x^2-2$. The equation $g(x) = x$ has two roots: $x = 2, -1$. The roots of $g(x) - x$ obviously are roots of $g(g(x)) - x$. Dividing $g(g(x)) - x$ by $g(x) - x$ and determining the roots of the resulting quadratic polynomial, we see that $g(g(x)) - x$ has roots

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = \frac{-1+\sqrt{5}}{2}, \quad x_4 = \frac{-1-\sqrt{5}}{2}.$$

Suppose $i \in \{1, 2, 3, 4\}$. Consider the sequence $x_i, f(x_i), f(f(x_i)), \dots$. It has a period 4 (maybe not the least one), thus all its terms are roots of $g(g(x)) - x$. Since $x_{1,2}$ are roots of $g(x)-x$, for $i = 1$ or for $i = 2$ the sequence has the period 2 as well, hence the sequence contains x_1 and x_2 only. For $i = 3, 4$ the number 4 is the least period because $x_{3,4}$ are not roots of $g(x) - x$. Thus in the sequence $x_3, f(x_3), f(f(x_3)$ all three numbers are distinct, so one of them equals x_1 or x_2. But then the above implies that all subsequent terms equal x_1 or x_2 while the fifth term must equal x_3.

3. Find all functions $f : R^2 \to R$ which satisfy the functional equation

$$f(\dots f(f(x_1, x_2), x_3) \dots, x_{2006}) = x_1 + x_2 + x_3 + \dots + x_{2006}.$$

Solution. $f(x, y) = f(0 + 0 + \dots + 0 + x, y)$ (with 2005 zeroes). Hence

$$f(x, y) = f(f(\dots f(f(0, 0)) \dots, x)y) = f(0, 0) + 0 + \dots + 0 + x + y = x + y + \text{const}.$$

Substituting this formula in the original equation, we get const $= 0$, that is, $f(x, y) = x+y$.

C) The result of solving a functional equation often depends on whether we impose the requirement of continuity on the functions sought. We shall not need the strict definition of continuity here. It suffices to know that any polynomial, exponent, logarithm (for $x > 0$), sine, cosine are continuous, and that any continuous function has the following properties.

(a) If the values of a continuous function at the points a and b are different then any intermediate value is achieved in some point of the interval $[a; b]$ (the Intermediate Value Theorem).

(b) If two continuous functions defined on the real axis coincide at all rational points then they coincide everywhere.

(c) If a continuous function is 1-1 then it is strictly monotonic. (The inverse is obvious.)

(d) A function continuous on a closed interval $[a; b]$ is bounded on this interval.

Monotonicity of a function also is sometimes important for solving functional equations.

4. If the function $f(x)$ is strictly monotonic, what can you say about the direction of growth of the function $f(f(x))$?

Answer. The function $f(f(x))$ is strictly increasing.

5. Does there exist a continuous function $f : R \to R$ such that $f(f(x))$ strictly decreases?

Solution. No. If $f(f(x))$ is strictly decreasing then it is 1-1. Then $f(x)$ is 1-1 as well and thus it is strictly monotonic by property (c). By the result of Problem 4, $f(f(x))$ strictly increases – a contradiction.

6. A continuous function $f(x)$ is such that $f(f(x)) = -x^2$ for all real x. Prove that $f(x) \leq 0$ for all real x.

Solution. In each of the domains $x \geq 0, x \leq 0$, the function $f(x)$ is 1-1 and hence strictly monotonic by property (c). According to the result of Problem 4, the sign of monotonicity is different for these two areas. Since $f(f(x))$ is not bounded from below, the same is true for $f(x)$. Hence $f(x)$ increases for $x \leq 0$ and decreases for $x \geq 0$ and thus has maximum at $x = 0$. It remains to observe that $f(0) = f(-0^2) = f(f(f(0))) = -f(0)^2 \leq 0$.

Supplementary questions for problem 6.

(a) Does there exist any function satisfying the conditions of the problem?

Answer. Yes: $f(x) = -|x|^{\sqrt{2}}$.

(b) Is the continuity condition essential here?

Answer. Yes: modify the function from (a), putting for example $f(2) = 2^{\sqrt{2}}$.

7. Does there exist a continuous function $f(x) : R \to R$ such that $f(f(x)) = x^2 - \frac{1}{2}$ for all real x? (Compare with Problem 2.)

Solution. No. Here we cannot apply the solution of Problem 2 (why?) but we may use method of solving Problem 6. We obtain that $f(x)$ must decrease for $x \leq 0$ and must increase for $x \geq 0$. Then by property (d) it is not bounded from above for $x > 0$. Suppose $f(0) = a$. Then a is the minimum value of $f(x)$, $a \neq 0$ and thus $-1/2 = f(a) > f(0) = a$. Hence there exists $b > 0$ such that $f(b) = -1/2$. But $b = f(c)$ for some $c \neq 0$ and so $c^2 - \frac{1}{2} = f(f(c)) = -\frac{1}{2}$, that is, $c = 0$ — a contradiction.

D) **8.** Find all continuous solutions for the additive Cauchy equation.

Solution. Put $c = f(1)$, find step by step $f(m)$, $f(m/n)$, $f(0)$ and $f(-m/n)$ for positive integers m, n and then apply property (b). We get $f(x) = cx$.

Which of the properties (a)–(d) of continuous functions has been used here? The method of finding continuous solutions of functional equations using this property is called the **Cauchy method**.

Answer: we have used property (b).

As is well known,

$$(xy)^n = x^n y^n, \quad \exp(x+y) = \exp(x)\exp(y) \qquad (x, y \in R)$$

and

$$\ln(|xy|) = \ln(|x|) + \ln(|y|) \qquad (x, y \in R \setminus \{0\}).$$

Using this facts and the result of Problem 8, solve the following problem.

9. Find all continuous solutions for the **Cauchy equations**:

(a) $$f(xy) = f(x) + f(y) \quad (x, y \in R \setminus \{0\});$$

(b) $$f(x+y) = f(xy) \quad (x, y \in R);$$

(b′) $$f(x+y) = f(x)f(y) \quad (x, y \in R).$$

(c) $$f(xy) = f(x)f(y) \quad (x, y \in R).$$

Solution. (a) To start with, let $x > 0$. Set $g(x) = f(e^x)$. Then

$$g(x + y) = f(e^{x+y}) = f(e^x e^y) = f(e^x) + f(e^y) = g(x) + g(y),$$

and so $g(x)$ satisfies the additive Cauchy equation. Since e^x and $f(x)$ are continuous, $g(x)$ also is continuous, and according to the result of Problem 8, the function $g(x)$ has the form $g(x) = cx$ where c is a constant. Then $f(x)$ is of the form $c \ln x$.

In particular, $f(1) = 0$. Putting $x = y = -1$, we obtain $f(1) = 2f(-1)$ and hence $f(-1) = 0$. For arbitrary $x < 0$, we obtain $f(x) = f(-x) + f(-1) = f(-x)$. Hence $f(x) = c \ln |x|$ for arbitrary $x \neq 0$.

(b) Put $y = 0$ and obtain $f(x) = f(0)$, so $f(x) \equiv$ const. Any constant obviously fits.

(b') If $f(x) = 0$ for some x then $f(z) = f(x)f(z - x) = 0$ for any z. Otherwise the function, being continuous, does not change its sign. Since $f(2x) = (f(x))^2$, this sign is positive and we may consider the continuous function $g(x) := \ln f(x)$. We have $g(x + y) = \ln(f(x)f(y)) = \ln f(x) + \ln f(y) = g(x) + g(y)$ and so the additive Cauchy equation is satisfied. Hence $g(x) = cx$ for some c, and $f(x) = e^{cx}$. Thus either $f(x) \equiv 0$ or $f(x) \equiv e^{cx}$.

(c) If $f(x) = 0$ for $x \neq 0$ then $f(z) = f(x \cdot x^{-1}z) = f(x)f(x^{-1}z) = 0$ for any z. Otherwise let $x > 0$. Then $x = z^2$ for some z, and $f(x) = f(z^2) = (f(z))^2 > 0$. Consider the function $\ln f(x)$. It is continuous and satisfies the additive Cauchy equation. According to the result of Problem 9a, our function is of the form $c \ln x$ for some constant c. Thus $f(x)$ for $x > 0$ has the form x^c.

Since $f(x)$ has to be continuous at zero, we have $c \geq 0$. For $x < 0$ we have:

$$f(x) = f((-1) \cdot (-x)) = f(-1)f(-x) = f(-1)(-x)^c.$$

At the same time, $1 = f(1) = f((-1) \cdot (-1)) = (f(-1))^2$, hence $f(-1) = \pm 1$. Both values do fit. Thus either $f(x) \equiv 0$, or $f(x) \equiv |x|^c$, or $f(x) \equiv x^c$ for $x \geq 0$ and $f(x) \equiv -|x|^c$ for $x < 0$ (where c is a non-negative constant).

A Hamel basis is a set of real numbers such that any real number has a unique representation of the form $r_1\alpha_1 + \ldots r_n\alpha_n$ where n is a positive integer, $r_1, \ldots, r_n$ are rationals, and $\alpha_1, \ldots, \alpha_n$ belong to the given Hamel basis. The existence of a Hamel basis is proved using the axiom of choice, and here we assume it as a given fact.

10. Do there exist any discontinuous solutions of the additive Cauchy equation? If yes, then how can the set of all its solutions be described? How can one describe the solutions of this equation that are non-negative for $x \geq 0$?

Solution. Let the values of the function on some Hamel basis by arbitrary. Extend the function by additivity. Non-negative solutions are of the form $f(x) = cx$ where $c \geq 0$. Indeed, functions of such form do fit. Conversely, if some solution $f(x)$ is not of this form then for some two numbers x, y from a Hamel basis we have

$$\frac{y}{x} = \alpha, \quad \frac{f(y)}{f(x)} = \beta < \alpha.$$

Suppose m/n is a rational number which is less than α and greater than β. Then $\frac{mn}{x} < \alpha x = y$ and

$$f(\frac{m}{n}x) = \frac{m}{n}f(x) > \beta f(x) = f(y)$$

and hence

$$f(y - \frac{m}{n}x) = (\beta - \frac{m}{n})f(x) < 0.$$

For comparison, now solve the following problem.

11.

$$f(x+y) = f^n(x) + f^n(y) \ (x, y \in R; \ n \text{ is a fixed positive integer}, \ n > 1).$$

Solution. Putting $x = y = 0$ we get $f(0) = 2f^n(0)$, hence $f(0)$ equals 0, $2^{-1/(n-1)}$ or (for odd $n > 1$) $-2^{-1/(n-1)}$. Constant functions with such values satisfy the original equation. Put now $y = 0$ for arbitrary x. We get $f(x) = f^n(x) + f^n(0)$. Thus $f(x)$ has not more than n distinct values. Now put $y = x$, x arbitrary. Taking the above into account, we get $f(2x) = 2f^n(x) = 2f(x) - 2f^n(0) = 2f(x) - f(0)$. Thus if $f(x) > f(0)$ then $f(x) < f(2x)$. Hence $f(2x) > f(0)$, $f(4x) > f(2x)$ etc., but f has only a finite number of values. The case $f(x) < f(0)$ is similar.

E) The **Pexider equation** is obtained from the additive Cauchy equation by replacing all occurences of f with different functions:

$$k(x+y) = g(x) + h(y).$$

12. (a) Solve the Pexider equation.

(b) Find all its continuous solutions.

Solution. (a) Denote

$$a = g(0), \quad b = h(0), \quad f(z) = -a - b + k(z).$$

Then $f(x+y) = f(x) + f(y)$, $k(x) = f(x) + a + b$, $g(x) = f(x) + a$, $h(x) = f(x) + b$. Take arbitrary values of $f(x)$ on some Hamel basis and extend it by additivity.

(b) In view of the result of Problem 8 and of (a), we have $k(z) = cx + a + b$, $g(x) = cx + a$, $h(x) = cx + b$ where a, b, c are arbitrary constants.