FUNCTIONAL EQUATIONS

Second stage

F) Suppose a function g satisfies the equation

$$g(x+y) = g(x) + g(y)$$

where the tuple (x, y) belongs to some subset Z of the plane R^2. If g can be extended to a function f satisfying the same equation for all $x, y \in R^2$ then we say that f is an *additive extension* of g.

13. Show that if Z is the unit square then any function g satisfying the Cauchy equation on Z has a unique additive extension to the whole plane.

Solution. An arbitrary number x may be represented as ny where $n \in N,\ y \in Z$. Set $g(x) = ng(y)$. Then correctness of the definition, additivity and uniqueness are easily checked.

14. Give an example of an infinite set Z and a function g which satisfies the Cauchy equation on Z but has no additive extension from Z to R^2.

Example. $Z = \{(x, \{x\}) \mid x \in R;\ g(x) = \{x\}\}$ where $\{x\}$ is the integer part of x.

The above results have an application, for instance, in the following economic-mathematical model described in [1], pp. 95–96.

15. Suppose we have to divide an amount S of money between $m > 2$ competing projects. Each of n experts makes a recommendation (expert j suggests to grant the project i with the sum ξ_{ij}), and finally the 'consensus' allocation is given by some function

$$\phi_i(\xi_{i1}, \ldots, \xi_{in}).$$

Observe that for each project the consensus allocation is determined by the sums recommended by the experts for this project only, but the form of this dependence may vary for different projects. We impose two natural requirements.

(i) If all experts allocate zero sum to some project then this project obtains 0 in the consensus allocation:

$$\phi_i(0, \ldots, 0) = 0 \quad (i = 1, \ldots, n).$$

b) If all the allocations recommended by the experts exhaust the sum S then this is true for the consensus allocation as well: $\sum\limits_{i=1}^{m} x_{ij} = S$ $(j = 1, \ldots, n)$ implies $\sum\limits_{i=1}^{m} \phi_i(x_{i1}, \ldots, x_{in}) = S$.

Show that under the above conditions, all the functions ϕ_i have the same (not depending on i) form $\sum \omega_j \xi_j$ where $\omega_j \geq 0,\ \sum \omega_j = 1$.

Solution. For brevity, denote $(S, \ldots, S)$ by $\mathbf{S}$ and $(x_{i1}, \ldots, x_{in})$ by $\mathbf{x_i}$. Observe that condition (ii) is equivalent to

$$\sum_{i=2}^{m} \phi_i(\mathbf{x_i}) + \phi_1(\mathbf{S} - \sum_{i=2}^{m} \mathbf{x_i}) = S.$$

For $\mathbf{x_2} = \ldots = \mathbf{x_n} = \mathbf{0}$ we deduce $\phi_1(\mathbf{S}) = S$ $(i = 1, \ldots, m)$. For $\mathbf{x_3} = \ldots = \mathbf{x_m} = \mathbf{0}$ we get $\phi_1(\mathbf{S} - \mathbf{x}) = S - \phi_2(\mathbf{x})$ for arbitrary $\mathbf{x}$. Putting $\mathbf{x_4} = \ldots = \mathbf{x_m} = \mathbf{0},\ \mathbf{x_2} = \mathbf{x},\ \mathbf{x_3} = \mathbf{y}$, we obtain the Pexider equation (for each coordinate):

$$\phi_2(\mathbf{x} + \mathbf{y}) = \phi_2(\mathbf{x}) + \phi_3(\mathbf{y})\ (\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y} \in [0; S]^n).$$

Put $\mathbf{x} = \mathbf{0}$. Then $\phi_2(\mathbf{y}) = \phi_3(\mathbf{y})$. Similarly we deduce that each ϕ_i equals the same function ϕ. We get the equation $\phi(\mathbf{x}+\mathbf{y}) = \phi(\mathbf{x})+\phi(\mathbf{y})$ for $\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y} \in [0; S]^n$. Obviously $\phi(\xi_1, \ldots, \xi_n) \equiv \phi(\xi_1, 0, \ldots, 0)+\phi(0, \xi_2, 0, \ldots)+\ldots+\phi(0, \ldots, 0, \xi_n) := f_1(\xi_1)+f_2(\xi_2)+\ldots+f_n(\xi_n)$ (here $\xi_1, \ldots, \xi_n$ are real variables). Each of functions $f_1, \ldots, f_n$ satisfies additive Cauchy equation on $[0; S]$. The sense of the problem implies non-negativity of values of $\phi, f_1, \ldots, f_n$. In view of the results of Problems 13 and 10, we see that $\phi(\mathbf{x})$ has the form $\phi(\mathbf{x}) = \sum\limits_{j=1}^{n} \omega_j \xi_j$ where $\omega_j \geq 0$. Since $\phi(\mathbf{S}) = S$, we have $\sum\limits_{j=1}^{n} \omega_j = 1$. Conversely, functions of this form satisfy the conditions of the problem.

G) **16.** Find all continuous real functions of a positive real variable which satisfy the equation

$$f(xy) = a(x) + b(x)c(y).$$

Answer. 1) $f(x) \equiv a(x) \equiv K$ (K is an arbitrary constant, $b(x) \equiv 0, c(y)$ an arbitrary continuous function.

2) $f(x) \equiv K_1, a(x) \equiv K_1 - b(x)K_2$ (K_1, K_2 are arbitrary constants), $b(x)$ an arbitrary continuous function, $c(y) \equiv K_2$.

3)

$$f(x) = K_1 \ln x + K_2, \quad a(x) = K_1 \ln x + K_2 - K_3K_4,$$

$$b(x) \equiv K_3, \quad c(y) = \frac{K_1 \ln y}{K_3} + K_4$$

(K_1, K_2, K_3, K_4 are arbitrary constants, $K_3 \neq 0$).

4)

$$f(x) = K_1(x^\alpha - 1) + K_2, \quad a(x) = K_1(x^\alpha - 1) + K_2 - K_3K_4x^\alpha,$$

$$b(x) = K_3x^\alpha, c(y) = \frac{K_1(x^\alpha - 1)}{K_3} + K_4$$

($K_1, K_2, K_3, K_4, \alpha$ are arbitrary constants, $K_3 \neq 0,\ \alpha \neq 0$).

Solution. Put

$$f_1(x) := f(x) - f(1),\ c_1(y) := c(y) - c(1),\ a_1(x) := a(x) - f(1) + b(x)c(1).$$

Then

$$f_1(xy) = a_1(x) + b(x)c_1(y), \tag{$'$}$$

$$f_1(1) = c_1(1) = 0. \tag{$''$}$$

Putting $y = 1$, we get $f_1(x) = a_1(x)$. If $a_1 \equiv 0$ then either b or c_1 is zero constant and we obtain classes 1 and 2 of functions in the answer (see above). Otherwise put $x = 1$. Then

$$f_1(y) \equiv b(1)c_1(y),$$

whence $b(1) \neq 0$. Put

$$b_1(x) := b(x)/b(1).$$

Then $f_1(xy) = f_1(x) + b_1(x)f_1(y)$, hence

$$f_1(xyz) = f_1(xy) + b_1(xy)f_1(z) = f_1(x) + b_1(x)f_1(y) + b_1(xyz)f_1(z),$$

$$f_1(xyz) = f_1(x) + b_1(x)f_1(yz) = f_1(x) + b_1(x)f_1(y) + b_1(x)b_1(y)f_1(z).$$

Compare these equations and take a value of z such that $f_1(z) \neq 0$. We get

$$b_1(xy) \equiv b_1(x)b_1(y).$$

Since $b_1(0) \neq 0$, we have, in view of the result of Problem 9(c), $b_1(x) = x^\alpha$ where α is an arbitrary constant. Thus $f_1(xy) = f_1(x) + x^\alpha f_1(y)$. If $a = 0$ then in view of the result of Problem 9(a) we obtain the class 3 of functions. Now suppose $\alpha \neq 0$. Take arbitrary $x, y \neq 1$. Since $f_1(xy) = f_1(x) + x^\alpha f_1(y)$, $f_1(xy) = f_1(y) + y^\alpha f_1(x)$, we have

$$\frac{f_1(x)}{x^\alpha - 1} := \frac{f_1(y)}{y^\alpha - 1}.$$

Since the left side does not depend on y and the right one on x, both are constants, and we obtain the class 4 of functions.

Many of you know that the integral of a power function is again a power function (with a coefficient) with the only exception: the integral of $1/x$ is the logarithm (all integrals are, of course, defined up to an additive constant). In a standard course of calculus, this fact is proved with the help of differentiation, and the cases of degree -1 and of all other degrees are treated separately.

17. Find the integral of x^a where x is a positive real variable, a an arbitrary constant, using the results of Problems 16, 9a and 9c as the base for your argument. It is not allowed to differentiate until you obtain the functional equation!

Solution. Suppose $g(x) = x^a$, $f(x)$ is the antiderivative for $g(x)$. We may assume $f(1) = 0$. Since $g(xy) \equiv g(x)g(y)$, we have

$$f(xy) = \int_1^{xy} g(t)dt = \int_1^x g(t) + \int_x^{xy} g(t)dt =$$

$$= f(x) + \int_1^y g(tx)d(tx) = f(x) + g(x)x \int_1^y g(t)dt =$$

$$= f(x) + x^{a+1} f(y).$$

We are in the conditions of Problem 16 with $f(x) = a(x) \neq \text{const}$, $b(x) = f(x)$, $c(y) = x^{a+1}$. If $a = -1$, we have case 3 with $K_2 = K_4 = 0$, $K_3 = 1$, so $f(x) = K_1 \ln x$. If $a \neq -1$, we get case 4 with $\alpha = a + 1$, $K_2 = K_4 = 0$, $K_3 = 1$, and so $f(x) = K_1(x^{a+1} - 1)$.

H) **18.** Find all continuous solutions of the **d'Alembert equation**

$$f(\phi + \psi) + f(\phi - \psi) = 2f(\phi)f(\psi)$$

under the condition $f(\pi/4) = \sqrt{2}/2$.

Solution. Putting $\psi = 0$, we get $f(0) = 1$. Then for some $C > 0$ we have $f(x) > 0$ for $x \in [0, C]$. Putting $\psi = \phi = x/2$, we get

$$(***) \qquad f(x) + 1 = 2f(x/2)^2.$$

Suppose $f(C) < 1$. Then $f(C) = \cos\alpha$ for some $\alpha \in [0; \pi/2)$. By $(***)$ we have $f(C/2) = \cos\alpha/2$, and using induction, we get $f(C/2^n) = \cos\alpha/2^n$ for all positive integers n. The original equation implies:

$$f\left(\frac{k+1}{2^n}C\right) = 2f\left(\frac{k}{2^n}C\right) f\left(\frac{1}{2^n}C\right) - f\left(\frac{k-1}{2^n}C\right) =$$

$$= 2\cos\left(\frac{k}{2^n}\alpha\right)\cos\left(\frac{\alpha}{2^n}\right) - \cos\left(\frac{k-1}{2^n}\alpha\right) = \cos\left(\frac{k+1}{2^n}\alpha\right).$$

By continuity (property (b)) $f(Cx) = \cos\alpha x$ for any x. Putting $c = \alpha/C$, $Cx = \phi$, we have $f(x) = \cos c\phi$. Condition $(**)$ implies $c = 8k \pm 1$.

Now suppose $f(C) > 1$. Then similar argument shows that $f(\phi) > 1$ for any ϕ, thus condition $(**)$ fails.

Comment. For $C > 1$, if we omit condition $(**)$, the equation has the solution $f(x) = \mathrm{ch}(cx)$ where c is an arbitrary constant, $\mathrm{ch}(x) := \dfrac{e^x + e^{-x}}{2}$.

19. Now can you present a functional equation defining

(a) the sine function $\sin x$?

(b) the tangent function $\tan x$?

Solution. (a) $f(\phi+\psi+\frac{\pi}{2}) + f(\phi-\psi+\frac{\pi}{2}) = 2f(\phi+\frac{\pi}{2})f(\psi+\frac{\pi}{2})$ under condition $(**)$.

(b) $f(x+y) = \dfrac{f(x)+f(y)}{1-f(x)f(y)}$ under condition, for instance, $f(\pi/4) = 1$.

$*$**20.** Using results of Problems 8 and 18, show that the vector addition in 3-dimensional Euclidean space is the only operation on pairs of such vectors which satisfies the following conditions:

(i) if both vectors are subject to the same rotation then the result of the operation also is subject to the same rotation;

(ii) the operation is commutative and associative;

(iii) two vectors pointing in the same direction yield a vector of the same direction whose length is the sum of the lengths of our initial vectors;

(iv) the sum of two vectors of equal length depends continuously on their angle.

The pattern of the solution (see [1], p. 13–18). Denote the operation under consideration by $\circ$ and call its result *the sum* of the vectors. Condition (iii) implies that $p \circ 0 = p$ for any vector p (0 is the zero vector). Applying (i), we have $-p \circ p = 0$. Taking (ii) into account, we obtain:

(v) for the operation $\circ$, vectors form an Abelian group whose neutral element is 0, and $-p$ is the inverse element for p.

Condition (i) and commutativity of $\circ$ imply that the sum of two vectors of equal length lies on the bissector of one of two angles between them. If the vectors have the same direction then this is the smaller angle by (iii). If this is the greater angle for some two vectors of equal length then by continuity (property (iv)) some two vectors of non-opposite direction have zero sum but this contradicts (v). Thus the sum always lies on the bissector of the smaller angle between vectors.

Fix now the angle φ between two vectors. If their length x is given then the length $g(x)$ of their sum is determined by (i). Denote the length of a vector v by $|v|$. Suppose two vectors p_1 and p_2 have the same direction as well as two vectors q_1 and q_2, and suppose the angle between p_1 and q_1 equals φ. If $|p_1| = |q_1| = x$ and $|p_2| = |q_2| = y$ then in view of (iii) we have $|p_1 \circ p_2| = |q_1 \circ q_2| = x + y$. We also have $|p_1 \circ q_1| = g(x)$, $|p_2 \circ q_2| = g(y)$. Then

$$g(x+y) = |(p_1 \circ p_2) \circ (q_1 \circ q_2)| = |p_1 \circ (p_2 \circ q_1) \circ q_2| =$$
$$= |p_1 \circ (q_1 \circ p_2) \circ q_2| = |(p_1 \circ q_1) \circ (p_2 \circ q_2)| = g(x) + g(y).$$

We have obtained the Cauchy equation for non-negative x, y which has non-negative solution. According to the result of Problem 10, $g(x) \equiv cx$ for some $c \geq 0$. In fact $c > 0$ since the sum of two nonzero vectors of non-opposite direction is not 0 (see above).

If the angle between two vectors of unit length equals 2ϕ then denote the length of their sum by $f(\phi)$. Suppose vectors $p_1.p_2, q_1, q_2$ of unit length are given, and the angles between p_1, p_2 and q_1, q_2 equal 2ψ, the angle between p_1, q_1 equals $2(\phi+\psi)$, and the angle between p_2, q_2 equals $2(\phi-\psi)$. Then the angle between $p_1 \circ p_2, q_1 \circ q_2$ equals 2ϕ. One can deduce that

$$f(\phi+\psi) + f(\phi-\psi) = 2f(\phi)f(\psi)$$

for $0 \leq \psi \leq \phi \leq \frac{\pi}{4}$. The function $f(\phi)$ is continuous by (iv), it equals 0 for $\phi = \frac{\pi}{2}$ and does not equal 0 for $0 < \phi < \frac{\pi}{2}$. The solution of Problem 18 implies that $f(\phi) \equiv \cos\phi$, and this in turn implies the assertion of the problem for vectors of equal length. It can be extended to the case of unequal length by means of geometrical argument not using functional equations; see [1], pp. 17–18.

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