Theorem on altitudes and the Jacobi identity

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The purpose of this problem set is to introduce a nice idea due to academician V. I. Arnold on a relationship between the theorem on altitudes of a triangle and the Jacobi identity [1].

First let us give an interesting application of the idea — a generalization of the theorem on altitudes to the case of the three-dimensional space:

Theorem on altitudes of a 'triside' [1]. Let a, b and c be pairwise non-parallel lines in 3-dimensional space. Let a', b' and c' be the three common perpendiculars to the pairs of lines b and c, c and a, a and b. Finally, let a'', b'' and c'' be the three common perpendiculars to the pairs of lines a and a', b and b', c and c' (it is given that the pairs are pairwise non-parallel). Then the lines a'', b'' and c'' have a common perpendicular (that is, there is a line crossing all of them and perpendicular to all of them).

0. Check that if the lines a, b and c belong to one plane then this theorem becomes to the theorem on altitudes of a triangle.

We start with an introduction to spherical geometry, in which the Arnold idea reveals most clearly.

Subject One. Spherical geometry and vector product.

Consider a unit sphere in three-dimensional space. By a *big circle* (*spherical line*) we mean a section of the sphere by a plane containing the center of the sphere. We identify the antipodes of the sphere.

To each point of the sphere we assign the vector looking from the center of the sphere to this point. By definition let us assume that all the vectors obtained from the given one by multiplication by a number (positive or negative) correspond to the same point.

To each spherical line we assign an arbitrary vector perpendicular to the plane containing the given line. Let us assume that all the vectors obtained from this one by multiplication by a number (positive or negative) correspond to the same line.

In what follows by a point we mean a point on the sphere, and by a line, a spherical line. The definitions of other natural objects of spherical geometry such as segment, triangle, perpendicular etc. are left as an exercise. The problems where we ask to find a geometrical sense of an object or an identity we do not suppose a unique answer, there can be a number of such interpretations.

We denote points by uppercase letters and lines by lowercase ones. We denote the vector corresponding to a point (line) by the same letter as the point (line) itself with the vector symbol.

If $\vec{A}$ and $\vec{B}$ are two vectors then by $[\vec{A}, \vec{B}]$ denote their vector product. Recall that *the vector product* of two vectors $\vec{A}$ and $\vec{B}$ is a vector perpendicular to both of the given vectors, with absolute value equal to the area of the parallelogram spanned by the vectors. The direction of this vector is given by the right hand law. By (A, B) denote the inner product of the vectors.

1. Let A and B be two points in the sphere. Prove that the vector $[\vec{A}, \vec{B}]$ corresponds to the spherical line passing through the points A and B.

2. Let a and b be two spherical lines. Prove that the vector $[\vec{a}, \vec{b}]$ corresponds to their intersection point.

3. Let A be a point and let b be a line. Prove that the vector $[\vec{A}, \vec{b}]$ (if nonzero) corresponds to the perpendicular dropped from the point A to the line b.

4. Let A, B and C be three points. What is the geometrical meaning of the condition $\vec{A} + \vec{B} + \vec{C} = 0$?

5. Let a, b and c be three lines. What is the meaning of the condition $\vec{a} + \vec{b} + \vec{c} = 0$?

6. Prove the identity $[\vec{A}, [\vec{B}, \vec{C}]] = \vec{B}(\vec{A}, \vec{C}) - \vec{C}(\vec{A}, \vec{B})$.

7. Prove *the Jacobi identity*:

$$[\vec{A}, [\vec{B}, \vec{C}]] + [\vec{B}, [\vec{C}, \vec{A}]] + [\vec{C}, [\vec{A}, \vec{B}]] = 0.$$

8. Let A, B and C be the vertices of a spherical triangle. What is the geometrical meaning of the Jacobi identity for the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$?

[1]This statement was suggested as a problem in a selection competition in the Depatment of Mechanics and Mathematics of Moscow State University for this year's international student contest

9. Prove that in Theorem on the altitudes of a 'triside' formulated above the lines a'', b'' and c'' are parallel to one plane.
10. Let A and B be two points. Let $\vec{A}$ и $\vec{B}$ be the *unit* vectors facing to these points from the center of the sphere. To which point does the vector $\vec{A}+\vec{B}$ correspond?
11. What is the geometrical meaning of the identity $[\vec{A}, \vec{B}+\vec{C}] + [\vec{B}, \vec{C}+\vec{A}] + [\vec{C}, \vec{A}+\vec{B}] = 0$?
12. Let A, B and C be three points in the sphere. Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be the unit vectors pointing from the center of the sphere to A, B and C. To which point does the vector $[\vec{A}, \vec{B}] + [\vec{B}, \vec{C}] + [\vec{C}, \vec{A}]$ correspond?
13. Let a and b be two lines. Let $\vec{a}$ and $\vec{b}$ be the corresponding vectors such that $|\vec{a}| = |\vec{b}| = 1$. What is the geometrical sense of the inner product $(\vec{a}, \vec{b})$ and the absolute value of the vector product $|[\vec{a}, \vec{b}]|$? Answer the similar question also for two points A and B; for a point A and a line b.
14. What is the geometric sense of the mixed product $([\vec{A}, \vec{B}], \vec{C})$ and the identity $(\vec{A}, [\vec{B}, \vec{C}]) = (\vec{B}, [\vec{A}, \vec{C}]) = (\vec{C}, [\vec{A}, \vec{B}])$?
15. Prove that in spherical geometry there are no distinct similar triangles — the angles of a triangle uniquely define the sides of the triangle.
16. Try to find the geometrical meaning of more algebraic objects and identities. Obtain the proofs of the corresponding theorems of spherical geometry (or, vice versa, the initial algebraic identities, if the corresponding geometrical theorems are already known).
17*. By *a circle* we mean a section of the sphere by an arbitrary plane. A point in the sphere and a spherical line are specific cases of a circle. Investigate how our correspondence between points and lines can be extended to include arbitrary circles, and thus obtain new theorems of spherical geometry. (We shall discuss this subject more after the intermediate consideration of the problems.)

Spherical geometry is an example of a non-Euclidean geometry. It is based on the same axioms as the Euclidean geometry except the axiom on parallel lines. In this geometry many theorems not using the fifth axiom remain true.

18*. Try to prove the theorems of spherical geometry you obtained above (for example, in problems 8 and 11) directly, in a purely geometrical way, starting from the axioms of geometry except the axiom on parallel lines.

The next subject is devoted to the proof of Theorem on altitudes of a 'triside'.

Subject Two. Lines in space and bivectors.

Fix a point O in the three-dimensional Euclidean space. To each line in space assign an ordered pair of vectors $(u; v)$ as follows: take two points A and B on the line and put $u = AB$, $v = [OA, OB]$. In the sequel an ordered pair of vectors is called *a bivector*.

19. Check that, up to multiplication of both vectors u and v by the same number, the constructed bivector does not depend on the choice of points A and B on the line.
20. Check that the initial line is uniquely defined by the constructed bivector.
21. Check that in this way one can obtain any bivector $(u; v)$ such that $u \neq 0$ and $v \perp u$.

Let us extend our correspondence between the lines and the bivectors to bivectors $(u; v)$ such that $v \not\perp u$. Denote by $\operatorname{pr} v$ the projection of the vector v to the plane perpendicular to the vector u. By definition we assume that the bivector $(u; v)$ corresponds to the same line as the bivector $(u; \operatorname{pr} v)$. We shall denote a bivector and the corresponding line by the same letter, and to distinguish them, we are going to put the symbol 'hat' ('$\hat{\ }$') above the notation of the bivector.

Let $\hat{a} = (u; v)$ and $\hat{b} = (u'; v')$ be two bivectors. Define their *sum* componentwise: $\hat{a} + \hat{b} = (u + u'; v + v')$. Define their *product* by the formula

$$[\hat{a}, \hat{b}] = ([u, u']; [u, v'] + [v, u']).$$

A sum and a product of two bivectors are again bivectors.

Remark*. This definition comes from the formula for the commutator in Lie algebra of the transformation group of the 3-space. One of the geometrical interpretations of our bivectors are *slide vectors* [3].

22. Let the lines a and b be non-parallel. Prove that the bivector $[\hat{a}, \hat{b}]$ corresponds to the common perpendicular to the lines a and b.

23. Let the lines a and b be non-parallel. Prove that the line corresponding to the bivector $\hat{a} + \hat{b}$ crosses the common perpendicular to the lines a and b.
24. Prove that the multiplication of bivectors satisfies the Jacobi identity.
25. Prove Theorem on altitudes of a 'triside'.
26. Try to obtain more stereometrical theorems this way.