Theorem on altitudes and the Jacobi identity

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In this subject we shall show how to use the Arnold idea to obtain theorems not only on points and lines but also on circles. Let us give one example:

Theorem on bisectors of a curved triangle. Let a, b and c be three pairwise intersecting circles in the plane. Draw a circle c' passing through both intersection points of the circles a and b such that the angle between c' and a is equal to the angle between c' and b. (There are two circles satisfying these conditions, choose the one having common points with the interior of the intersection of the areas bounded by a and b.) Define the circles a' and b' analogously. Then the three circles a', b' and c' have a common point.

27. Prove the theorem on bisectors of a curved triangle.
28*. Fix a circle I in the plane. By a *common perpendicular* to a pair of circles a and b we mean a circle perpendicular to all the three circles a, b and I. Prove that Theorem on altitudes of a 'triside' remains true, if one replaces the word 'line' (in space) by 'circle' (in the plane) everywhere in the statement.

Let us start again with spherical geometry, in which the Arnold idea is revealed most clearly. The problems of this subject can be solved independently from the others.

Subject Three. Summing of circles.

By a *circle* we mean the section of the sphere by an arbitrary plane, not necessarily passing trough the center of the sphere. To each circle assign a vector perpendicular to this plane, looking toward the plane and having length $1/h$, where h is the distance between the plane and the center of the sphere.

In what follows uppercase letters always denote circles. The vector corresponding to a circle is denoted by the same letter as the circle, and, to distinguish them, with the symbol of the vector. For a vector $\vec{A}$ denote by d_A the length of a tangent to our sphere, dropped from the end of the vector $\vec{A}$.

29. What is the geometric sense of the condition $\vec{A}+\vec{B}+\vec{C}=0$?
30. Prove the formula $(\vec{A},\vec{A})=1+d_A^2$.
31. What is the geometric sence of the condition $(\vec{A},\vec{B})=1$?
32. To what circle does the vector

$$\frac{[\vec{A},\vec{B}]+[\vec{B},\vec{C}]+[\vec{C},\vec{A}]}{(\vec{A},\vec{B},\vec{C})}$$

correspond?
33. What is the geometric sense of the condition $\vec{A}-\vec{B}+\vec{C}-\vec{D}=0$?
34. To what circle does the vector

$$\frac{d_B}{d_B-d_A}\vec{A}-\frac{d_A}{d_B-d_A}\vec{B}$$

correspond?
35. Use these facts and algebraic identities to obtain more geometric theorems!

Appendix. Presentation of circles by points in space.

Take a circle in the plane be given by the equation $x^2+y^2+ax+by+c=0$. To this circle we assign the point in space having the coordinates (a,b,c).

36. For which numbers a, b and c there is a circle corresponding to the point (a,b,c)?
37. Consider a circle corresponding to the point P in space. What is the set of all the points corresponding to the circles orthogonal to our one?
38. What pairs of points correspond to a pair of
a)intersecting;
b)non-intersecting;
c)tangent circles?
39. Prove the following theorem (V. Yu. Protasov).
Take three circles such that the first is inside the second and the second is inside the third. Consider chains of circles tangent to both the first and the third one and such that one of the intersection points of any two consequent circles in the chain belong to the second given circle. If this chain is closed for some starting circle, then it is closed for any choice of starting circle.
40. Take a sphere, a plane tangent to the sphere and four points A, B, C and D in the plane. Let D' be the intersection point of three planes passing through the lines AB, BC and CA respectively and tangent to the sphere. Define the points A', B' and C' analogously. Prove that the points A', B', C' and D' lie in one plane tangent to the sphere.