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Around of Feet of Bisectors

Introduction

Some theoretical facts

- The Euler line.
- The nine-point circle.
- Orthocentric quadruple. Some properties of the orthocenter.
- The incircle and the excircle, their centers.
- Power of a point with respect to a circle, the radical axe of two circles, the radical center of three circles.

Problems

1. Mansion lemma in common case. *The middle of arc AC of circumcircle of triangle ABC non-containing of vertex B is equidistance from vertices A and C, center I of its incircle and center I_2 of its B-excircle. The middle of arc AC of circumcircle of triangle ABC containing of vertex B is equidistance from vertices A and C, centers I_1 and I_3 of its A-excircle and C-excircle.*
2. Euler formula for incircle and excircle. *Distance between the centers of the incircle and the circumcircle satisfies to formula: $IO^2 = R^2 - 2Rr$. Distance between the centers of the excircle and the circumcircle satisfies to formula:* $I_kO^2 = R^2 + 2Rr_k$, $k = 1,2,3$.
3. Poncelet theorem (internal). *If Ω and ω are the circumcircle and the incircle of any triangle respectively then there are endless amount of triangles with the same circumcircle and incircle and any point of Ω may be the vertex of such triangle.* Prove that condition of Poncelet theorem is equivalent to Euler formula for the incircle and the circumcircle.
4. Poncelet theorem (external). *If Ω and ω_1 are the circumcircle and the excircle of any triangle respectively then there are endless amount of triangles with the same circumcircle and excircle.* Prove that the given condition of Poncelet theorem is equivalent to Euler formula for the excircle and the circumcircle.
5. The feet of external bisectors are collinear. (This line is called *the axe of external bisectors*, we note it ℓ). Line ℓ is perpendicular to line IO.
6. Express distance from point I to the axe of external bisectors through the radii of the incircle and the circumcircle.
7. Let A_1, B_1 и C_1 be the feet of internal bisectors on respective sides of triangle. Lines $A_1B_1 = \ell_3$, $B_1C_1 = \ell_1$, $C_1A_1 = \ell_2$ are called the axes of internal bisectors. Prove that each of these lines passes through the foot of the respective external bisector and that ℓ_k is perpendicular to line I_kO (k = 1, 2, 3).

8. Geometrical analog of external Euler formula. *Let circles* Ω *u* ω_1 *intersect in points* P *and* Q. *Then* Ω *is circumcircle and* ω_1 *is excircle for a triangle if and only if tangents to* ω_1 *at points* P *and* Q *secondary intersect* Ω *in tangent points of common extangents to* Ω *and* ω_1.
9. Let Ω and ω be the circumcircle and incircle for a triangle. Then the locus of feet of external bisectors of family of triangles with the same circumcircle and incircle is a line.
10. Let Ω and ω_1 be the circumcircle and A-excircle for a triangle, P and Q are the tangent points of Ω and common extangents of these circles. Then the locus of feet of internal bisectors of angles *B* and *C* of family of triangles with the same circumcircle and A-excircle is interval *PQ* (without *P* and *Q*).
11. Circumradius of a triangle equals to exradius if and only if the circumcenter lies on respective axe of internal bisectors.
12. Let B_0' be the middle of arc AC of Ω containing B, B_2 be the foot of external bisector on AC. Prove that $I_2B_0' \perp B_2I$.