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Around the Feet of Bisectors

Main problems

Theory

- Nagel point.
- Gergonne point.
- Isogonality.
- Symmedians and Lemoine point.
- Quadrilateral complete. The Gauss line and the Aubert line.
- Three homothety centers theorem.
- Feuerbach theorem. Feuerbach points

13. Let the triangle ABC be given. The bisector of angle A intersects the sideline BC in point A_1 and the circumcircle in point A_0. The points C_1 and C_0 are defined similarly. B_{00} is the common point of lines A_0C_0 and A_1C_1. Then the lines $B_{00}I$ and AC are parallel.
14. Prove that the line $B_{00}B$ touches the circumcircle of *ABC*.
15. Let the circle b_{00} with center B_{00} pass through B. Then b_{00} passes through I.
16. Let B_3 be the common points of circle Ω, distinct from B, with b_{00}. Let B_0 be the common points of circle Ω, distinct from B, with the bisector of angle B. Then $\angle B_0B_3I = 90^\circ$.
17. Let N be the Nagel point of ABC. Then the lines BN and BB_3 are isogonal with respect to *ABC*.
18. Let L be the common point of A_0C_1 and C_0A_1. Then the midpoint of the segment AC lies in the line LI.
19. Let the points A_{00} and C_{00} be defined similarly to B_{00} (see problem 13). Then these three points are collinear. Let the line $A_{00}B_{00}$ be denoted by l_{00}. Then l_{00} is parallel to the external bisectors axe.
20. Let the bisectors of triangle *ABC* intersect its circumcircle in points A_0, B_0, C_0, distinct from *A, B, C*. Then the common points of lines AB and A_0B_0, BC and B_0C_0, CA and C_0A_0 are collinear on the line l_0 parallel to external bisectors axe.
21. Prove that the lines ℓ_{00} and ℓ_0 divide the distance between I and the external bisectors axe ℓ into three equal parts.
22. Let the internal bisectors axe ℓ_2 intersect Ω in the points E, D. The circle passing through the points I, E, D is called b_2. Then b_2 passes through the excenters I_1 and I_3.

23. Prove that the radius of b_2 is twice greater than the circumradius *R*.

24. Prove that I_2 is the homothety center of Ω and b_2.

25. Prove that the line $B_{00}I$ (see problem 13) is tangent to the circle b_2.

26. Prove that the tangents to the excircle ω_2 in its common points with Ω are also tangent to the circle b_2.

27. Let D be one of common points of internal bisectors axe ℓ_2 with Ω. D' is the point of Ω opposite to D. Then one of common points of Ω and ω_2 lies on the line $D'I_2$ and the second lies on the circle $(OD'I_2)$.

28. Given the radius R and r_2. Find the distance between the common points of internal bisectors axe ℓ_2 with Ω.

29. Prove that the line OB touches the circle passing through *B* and the feet of the internal and external bisectors of angle *ABC.*

30. Let *K* be distinct from *B* common point of Ω with the circle passing through *B* and the feet of the internal and external bisectors of angle *ABC.* Prove that *K* lies on the symmedian of angle ABC.

31. Prove that the circumcenter of the triangle lies in the Aubert line of quadrilateral formed by four bisectors axes.

32. Prove that the Lemoine point of the triangle lies in the Ober line of quadrilateral formed by four bisectors axes. (Remark: from the problems *31 and 32 the orthocenter of the triangle formed by the feet of internal bisectors is collinear with the circumcenter and the Lemoine point.*)

33. Let B_5 be the common point of tangents defined in problem 26. Prove that B_5 lies in the segment between I and the touching point of ω_2 with the sideline AC.

34. Let the points A_5 and C_5 be defined similarly to B_5. Then the lines AA_5, BB_5, CC_5 are concurrent in the point T. The point T is collinear with Gergonne point G and the centroid M and $GM:MT = 2:1$.

35. Let F, F_1, F_2, F_3 be the internal and external Feuerbach points. Prove that the feet of internal and external bisectors lies in 6 lines defined by F, F_1, F_2, F_3.

36. (*V.Tebault*) Prove that the triangles formed by the feet of external bisectors $(\Delta A_1B_1C_1)$ and external Feuerbach points $(\Delta F_1F_2F_3)$ are similar.

37. Prove that the circumcircle of the triangle formed by the feet of external bisectors $(\Delta A_1B_1C_1)$ pass through the internal Feuerbach point.