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Around the Feet of Bisectors

Main problems

Solutions

Using solutions by Bazhov I. and Chekalkin S.

13. The external bisector BB_2 (fig. 13) is parallel to A_0C_0, hence $\frac{C_1B_{00}}{B_{00}B_2} = \frac{C_1X}{XB}$. In the triangle XBY the line BI is the bisector and the altitude, it means that BI is the median. Because $C_0I = C_0B$ A_0C_0 (i.e. XY) is the perpendicular bisector of BI. Therefore $IX \parallel BC$. Hence $\frac{C_1X}{XB} = \frac{C_1I}{IC} \Rightarrow \frac{C_1I}{IC} = \frac{C_1B_{00}}{B_{00}B_2}$, i.e. $IB_{00} \parallel AC$.

14. Because B_{00} lies on the perpendicular bisector of BI (fig. 13), $\angle B_{00}BI = \angle B_{00}IB \Rightarrow \angle B_{00}IB = \angle B_2B_1 = \frac{1}{2}\overset{\cup}{BB_0}$. It means that $\angle B_{00}BB_0 = \frac{1}{2}\overset{\cup}{BB_0} \Rightarrow B_{00}B$ is tangent to Ω.

15. Because B_{00} lies on the perpendicular bisector of BI (fig. 13), $B_{00}I = B_{00}B$.

16. $\angle IB_3B_0 = \angle B_0B_3B' + \angle B'B_3I = \angle B'BB_0 + \angle B'B_3I$ (fig. 16). Because $\angle B_0BO = 90^\circ$, $B'B$ is tangent and $\angle B'BI = \frac{1}{2}\overset{\cup}{BI}$. Then $\angle B_0B_3I = \angle B'B_3I + \angle IB_3B = \angle B'B_3B = 90^\circ$.

17. Let us consider the homothety ω and ω_2 with the center B (fig. 17). It is not difficult to show that BN intersects ω in the point opposite to the touching point of ω and AC. This means that the common point of lines BN и B_0B_0' is such point K that $IK \parallel AC$.
Now as $\angle IBB_0' = \angle IKB_0'$ the quadrilateral $IBB_0'K$ is cyclic (fig. 17). By problem 16 the points B_0', I and B_3 are collinear. So $\angle IBN = \angle IBK = \angle IB_0'K = \angle B_3B_0'B_0 = \angle B_3BB_0$.

18. As the quadruple of rays IA_0, IL, IC_0, IB_{00} is harmonic (fig. 18) the quadruple IA, IM, IC, IB_{00} also is harmonic. But by problem 13 $IB_{00} \parallel AC$ so M is the midpoint of AC.

19. Consider the homothety with center C_1 transforming I to C (fig. 19). As $IB_{00} \parallel B_2C$, this homothety transforms $B_{00} \to B_2$, $A_{00} \to A_2$. It follows that $A_{00}B_{00} \parallel A_2B_2$. Similarly $B_{00}C_{00} \parallel B_2C_2$. So A_{00}, B_{00}, C_{00} are collinear on the line parallel to ℓ.

20. Consider the homothety with center C_0 transforming I to C. As $IB_{00} \parallel B_2'C$ this homothety transforms $B_{00} \to B_2'$, $A_{00} \to A_2'$. It follows that $A_2'B_2' \parallel A_{00}B_{00} \parallel A_2B_2$. Similarly $B_2'C_2' \parallel B_2C_2$. So A_2', B_2', C_2' are collinear on the line parallel to ℓ.

21. Consider the perpendiculars to BI passing through B, I and the midpoint of the segment BI (fig. 21). As $B_{00}I = B_{00}B$ and $B_{00}B$ touches the circle Ω, the degrees of B_{00} with respect to Ω and the point I are equal. As $\angle B_6IA = \angle BIA - 90^\circ = \angle ICA$, $B_6I^2 = B_6A \cdot B_6C$, the radical axis of Ω and I coinciding with ℓ_{00} passes through B_6. The triangle B_8BI is recanguler so $B_{00}B = B_{00}I = B_{00}B_8$. As three considered lines are parallel and the lines $B_{00}B_6$ and B_7B_2 are also parallel, $B_{00}I = B_4B_6 = B_{00}B_8 = B_2B_4 = B_7B_8$. So the segment IB_7 is divided by ℓ_0 and ℓ_{00} to three equal parts.

22. By problem 7 ℓ_2 is the radical axis of the circles Ω and (II_1I_3). So the degrees of bisector's feet A_1 and C_1 with respect to these circles are equal. This means that the points D and E lying in this radical axis are the common points of Ω and (II_1I_3).

23. The circle Ω is the nine-point circle of the triangle II_1I_3 and the circle b_2 is its circumcircle. So the ratio of their radius is equal to 2.

24. As I_2 is the orthocenter of the triangle II_1I_3 the circles Ω and b_2 are homothetic with center I_2.

25. The radical axis of circles Ω и b_2 coincide with the line DE and passes through the point B_{00}. As $B_{00}I = B_{00}B$ the segment $B_{00}I$ is the tangent to the circle b_2.

26. The homothety with center I_2 and coefficient $\frac{1}{2}$ transforms the point D to the point D_1 lying on Ω (fig 26). As PD touches ω_2 $\angle DPI_2 = 90^\circ$. It means that $PD_1 = DD_1$, so the tangent to Ω in D_1 is parallel to PD. The homothety with center I_2 and coefficient 2 transforms this tangent to the line PD. But this homothety transforms Ω in b_2. So PD touches b_2.

27. As DP touches ω_2 and DD' is the diameter of Ω, $\angle DPI_2 = \angle DPD' = 90^\circ$. It follows that the points D', P and I_2 are collinear (fig. 27). Now $\angle OD'I_2 = \angle OPD' = 180^\circ - \angle OPI_2 = 180^\circ - \angle OQI_2$. It means that the points O, D', I_2 and Q are on the circle.

28. The chord DE is twice greater than the segment DL (fig 28). But

$$DL = OD \cdot \sin\angle I_2OD = R \cdot \frac{KI_2}{OI_2} = R \cdot \frac{DS}{\sqrt{R^2 + 2R \cdot r_2}} = \frac{R}{\sqrt{R^2 + 2R \cdot r_2^2}} \cdot \sqrt{OI_2^2 - OK^2} =$$
$$= R \cdot \sqrt{\frac{4Rr_2 - r_2^2}{R^2 + 2R \cdot r_2^2}}.$$

So $DE = 2\sqrt{R \cdot r_2 \cdot \dfrac{4R - r_2}{R + 2r_2}}$.

Remark: Using this formula it is easy to prove that $r_2 = R$ follows $DE = 2R$ i.e. DE is the diameter of Ω (problem 11). Also the inequality for the exradius can be obtained

29. As $\angle BB_1B_2$ is the external angle of the triangle ABB_1 (fig. 29) $\angle BB_1B_2 = \angle A + \angle B/2$. from the right-angled triangle B_1B_2B we obtain that $\angle BB_2B_1 = 90^\circ - \angle A - \angle B/2$. On the other hand, $\angle OBB_1 = \angle OBC - \angle B_1BC$, so $\angle OBB_1 = (90^\circ - \angle A) - \angle B/2$. Thus $\angle BB_1B_2 = \angle OBB_1$, that is, OB touches the circle (B_1BB_2).

30. As (B_1BB_2) is the Apollonius circle of the segment AC (fig. 30) we have $\dfrac{AB}{BC} = \dfrac{AK}{KC}$. Thus $AB \cdot KC = AK \cdot BC$ and the quadrilateral $ABCK$ is harmonic. It follows that the tangents to Ω at its vertices A and C intersect in the point S lying in the line BK. It means that BK is the symmedian of the triangle ABC.

31. It is well known that the Aubert line of the complete quadrilateral is the radical axis of three circles having its diagonals as diameters. If the triangle is formed by the bisectors axis than the segments A_1A_2, B_1B_2 и C_1C_2 are its diagonals. By problem 30 the degrees of the point O with respect to three circles having this segments as diameters are equal to R^2. So O lies in the radical axis of these circles.

32. The degree of Lemoine point L with respect to the circle having B_1B_2 as diameter is equal to $-BL \cdot LK$ (fig.30). The degree of this point with respect to Ω is the same. So the degrees of L with respect to three circles having the diagonals of the quadrilateral as diameters are equal and L lies on the Aubert line.

33. By problem 26 B_5 is the common point of two common internal tangents of the circles ω_2 and b_2. So B_5 is the homothety center of these circles. This homothety transforms the line AC touching ω_2 to the parallel line IB_{00} touching b_2. So the touching point of ω_2 with the sideline AC and the incenter I are the respective points of this homothety. It means that the point B_5 lies in the segment between these points

34. *The authors don't know the synthetic solution of this problem. The participants also didn't find it. There is a plan of calculation in barycentric coordinates.*
Using the coordinates of the Gergonne point G and the centroid M we can find the coordinates of the point T dividing the segment GM in the ratio equal to -3. Also we can find the ratio in which the line BT divide the sideline AC. Now as B_5 divide the segment IB'' (B'' is the touching point of ω_2 with AC) in the ratio equal to $2R/r_2$ (problem 33), we can find the coordinates of B_5 and verify that the lines BB_5 and BT divide AC in the equal ratio.

35. The foot of internal bisector is the internal homothety center of the incircle and the respective excircle. The foot of the external bisector is the external homothety center of two excircles. The point F is the external homothety center of the incircle and the nine-point circle and the points F_1, F_2 и F_3 are the internal homothety centers of the nine-point circle and respective excircles. The asertion of the problem follows from three homothety centers theorem.

36. The solution is in the book: I. F. Sharygin. «Geometry 9-11», problem № 586 (in Russian).

37. By three homothety centers theorem the points F_2, B_1 and F are collinear. Similarly the points F_3, C_1, F are collinear and the points F_1, A_1, F are collinear. Thus the triangles $F_1F_2F_3$ and $A_1B_1C_1$ are perspective with center F. Also by problem 36 these triangles are similar. So the sum of the angle $C_1B_1A_1 = F_1F_2F_3$ and the angle C_1FA_1 is equal to 180°. It means that the points A_1, B_1, C_1, F are concyclic.