Coverings by the cell figures.
(Terminator-3 is back at Sorrento...)

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1 The aims review and the notation.

We consider the *cellular line*, which is an infinite strip divided into the square cells (this strip is one cell wide). In some problems we also consider the cellular plane and the cellular space (that is, the space divided in a standard way into the unit cubes). If you want, you also can work in a cell n-dimensional space. Nevertheless, it is sufficient to consider only cell line and cell plane for all essential problems.

Definition 1. *A* figure *is any finite set of cells (this set is not supposed to be connected!). Clearly, the* area *of a figure is a number of its cells. Any translation of a figure (by the vector with integer coordinates) is called a* translate *of this figure. Let $\mathcal{F}$ be a figure. A family of translates of a figure $\mathcal{F}$ is called an $\mathcal{F}$-*covering *of a line (plane, etc.), if each cell of this line (plane, etc.) is covered by at least one of these translates.*

Enumerating all the cells of a cellular line, we identify it with the set $\mathbb{Z}$ of integer numbers. Similarly, we identify the cellular plane with $\mathbb{Z}^2$ and so on. This way, the translate of a one-dimensional figure $\mathcal{F}$ by the number t is a figure $\mathcal{F}+t=\{x+t \mid x\in\mathcal{F}\}$, and the covering $\{\mathcal{F}+t_i\}$ is determined by the set $T=\{t_i\}$. Thus, we say that T *determines* a covering if $\mathbb{Z}=T+\mathcal{F}(=\{t+f \mid t\in T,\, f\in\mathcal{F}\})$. If $0\in\mathcal{F}$, then we can simply say that T is a set of cells which are covered by the images of 0 under the parallel translations considered.

All these notions can be introduced for the $\mathbb{Z}^k$ (with replacing numbers by the integral vectors).

This problem is devoted to the following **Question.** *Given a figure $\mathcal{F}$. How to find (or to approach) the "most efficient" covering of a line by $\mathcal{F}$?*

First, we should give a strict definition of the "efficiency" of a covering. Suppose that a segment $[1,N]$ is covered by d translates of a figure $\mathcal{F}$ having the area n. Then it is natural to define the *efficiency* of this covering as $N/(dn)$. For an infinite strip, we have to tend to the limit.

Definition 2. *Consider a set $S\subset\mathbb{Z}$. For an arbitrary N, introduce the set $S_N=S\cap[-N,N]$. If a sequence $S_N/(2N)$ has a limit $\rho(S)$, then we call this limit the* density *of the set S.*

Now suppose that a set T determines the covering by a figure $\mathcal{F}$, $|\mathcal{F}|=n$. Then we define the non-efficiency *of this covering as $\rho(T)n$ (of course, if it exists).*

Imagine now that each cell of each translate is a brick. If a cellular line is covered by these translate, then there appears a column of bricks on each cell. Informally speaking, the non-efficiency average height of such a column, or the average number of layers in our covering.

Exercise for an understanding. Give the strict definition of the notion of "average number of layers" and prove that it coincides with the non-efficiency.

It is very useful to understand that the sets S and $S+n$ have or have not the density simultaneously; moreover, in the former case $\nu(S+n)=\nu(S)$. Actually, analogous definitions are introduced for $\mathbb{Z}^k$; in this case, one should of course take the intersection of the set S with the cube $[-N,N]^k$.

Now, we introduce the main notions in our problem.

Definition 3. *A* non-efficiency *of a figure is the limit*

$$\nu(\mathcal{F}) = \inf_{T:T+\mathcal{F}=\mathbb{Z}^k} |\mathcal{F}|\rho(T),$$

where the infimum is taken over all possible coverings by the figure $\mathcal{F}$. Introduce also

$$a_1(n) = \sup_{\mathcal{F}\subset\mathbb{Z}:|\mathcal{F}|=n} \nu(\mathcal{F}), \quad a_2(n) = \sup_{\mathcal{F}\subset\mathbb{Z}^2:|\mathcal{F}|=n} \nu(\mathcal{F}), \quad \dots$$

Thus, the non-efficiency of the figure is a "minimal" non-efficiency of a covering by this figure; $a_1(n)$ is a "maximal" non-efficiency of the figure consisting of n cells on a cellular line ($a_2(n)$ is a similar notion for the cellular plane, and so on).

2 Introductive problems.

2.1. The non-efficiency of each figure of area 2 is 1.

2.2. ***(i)*** Find the non-efficiency of the figure ■■□■;
(ii) Estimate the non-efficiency of the figure ■□■□□□■□■;
(iii) and ■□■□□□■□■□□□□□□□□□■□■□□□■□■;
(iv) and even ■■■■□□□□■■□□■■□□■□■□■□■□;
(v) find another figure for the training if you wish.

2.3. Prove that $a_1(n) \le a_1(2n)$

2.4. Prove that the non-efficiency of a figure is 1, if and only if there is a tiling by this figure
(i) on the line;
(ii) in the plane.

2.5. Consider the figures of area 3 in the cellular line.
(i) Find such a figure of non-efficiency 6/5.
(ii) Prove that the non-efficiency of any figure of the area 3 is not more than 6/5.

2.6. Let $\mathcal{F}$ be a figure of area 4.
(i) Prove that $\nu(\mathcal{F}) \le 8/5$.
(ii) Determine whether this estimate is sharp.
(iii) Present an example of a figure $\mathcal{F}$ having the area 4 and non-efficiency $\nu(\mathcal{F}) \ge 3/2$.

3 The main results

3.1. On the line, given a figure of non-efficiency α.
(i) Prove that there exists a covering with the non-efficiency α.
(ii) Prove that there exists even a periodic covering with the same non-efficiency (and therefore α is a rational number).

3.2.
(i) Prove that $a_1(n) = a_2(n)$. (Hence, in the following text we denote it by $a(n)$.)
(ii) $\dots = a_3(n) = a_4(n) = \dots$.

In the following, you can use the latter result without the proof if necessary.
3.3. Prove that $a(n) \to \infty$ as $n \to \infty$.

3.4. ***(i)*** Denote by $\aleph(n)$ the maximal possible non-efficiency of the figure having area n and diameter $\le 2n$. Prove that $\aleph(n) \le c \ln n$ for some constant c
(ii) Prove the similar result for the figures of diameter $\le kn$ ($k > 1$ is an arbitrary constant).
(iii) It follows from the next problem that there exists even a common constant c for all values of k. Is it possible to achieve the better result for a fixed k? (The authors do not know the answer!)

3.5. ***(i)*** Prove that $a(n)/n \to 0$ as $n \to \infty$. *The authors do not know the solution which is simpler than the one for the following:*

(ii) Prove that $a(n) \leq c \ln n$ for some constant c.

3.6. Find as best as possible lower bound for $a(n)$. It seems to the authors that they know the estimate of the form $a(n) \geq c \ln n / \ln \ln n$ for some constant c.

4 Open questions

Will my head succeed in running away
from the wave's enormous throat?
And if it does, what will be the price?
But if it doesn't, why so?

M. Scherbakov,
transl. by L. Schulz

Here, the questions for the further investigation are presented.

4.1. Is it true that the sequence $a(n)$ is non-decreasing?

4.2. Given a figure $\mathcal{F}$, let $b(\mathcal{F})$ be the maximal possible density of the family of non-intersecting translates of this figure. Try to find some joint estimates for $a(\mathcal{F})$ and $b(\mathcal{F})$. For instance, it is interesting whether it is true that $a(\mathcal{F})b(\mathcal{F}) \leq 1$ for every figure $\mathcal{F}$.

The next question may help to find some upper bounds for $\nu(\mathcal{F})$.

4.3. For any n, find (or estimate) the maximal $k = k(n)$ such that it is possible to cut a tile with area k from any figure with area n. (A figure is a *tile* if there exists a tiling by tyhe translates of this figure.)

4.4. Try to estimate the *maximal* (instead of an average one!) number of layers in an optimal (by this parameter) covering by the figure given.