Auxiliary conics

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Solutions

1. Using the theorem 3 for four given lines and infinity line we receive that tere exists the single parabola touching the given lines. By the theorem 1 its focus is the Michel point of the quadrilateral and the directrix is its Aubert line. Also we can apply the dual theorem 10 to for given lines, the infinity line and three medial lines of the diagonal triangle. It results that the medial lines touche the same parabola, so we receive the proof of p.c).

2. Let tehe conic $ABCPQ$ be considered. The pole of PQ with respect to this conic is the sought points. This follows from the p.c) of the addition and the theorem 4.

Addition

a) the incenter;

b) the Lemoine point isogonally conjugated to the centroid;

c) the infinite point with the direction perpendicular to PQ.

3. Let X' be the projection of X to the directrix. Suppose that the bissector of angle FXX' don't coincide with the tangent. Then it intersect the parabola in some point Y distinct from X and $FY = YY'$, where Y' is the projection of Y to the directrix. As $FX = XX'$, $\angle FXY = \angle X'XY$ the triangles FXY and $X'XY$ are congruent, so $FY = YX'$. But $YY' < YX'$ — contradiction.

4. Using the notations of previous problem we receive that F and X' are symmetric with respect to the tangent.

5. Using the result of previous problem we receive that the sought locus is the tangent to the parabola in its vertex.

6. Let the pencil of conics passing through the incenter and the excenters of ABC be considered. By the theorem 8 all polars of X with respect to the conics of this pencil pass through X'. Let $\mathcal{C}$ be such conic that the polar of X with respest to $\mathcal{C}$ coincide with $X'Y'$. Then the polar of Y' pass through X, and so coincide with XY. As U is the common point of polars of X and Y', XY' is the polar of U. So the polar of U pass through V. Similarly we can find such conic that the polar of U coincide with $X'Y$. By the theorem 8 V is isogonally conjugated to U.

7. By the Brianchon theorem the given triangles have cjmmjn inscribed conic. Let some line passing through A' be intersect BC in the point P, and the parallel line passing through B' be intersect AC in the point Q. Using the Brianchon theorem for the hexagon $A'XQPYB'$ where X, Y are the the infinite points of AC and BC repectively, we obtain that the line PQ also touches this conic. This follows the sought assertion.

8. The common points of common external tangents are collinear. So we can use the dual four conics theorem.

9. By the theorem 7 T is the center of some inscribed conic. As the distances from T to opposite sidelines are equal, these sideline are parallel or symmetric with respect to some axis of this conic. If the both pairs of opposite sideline are symmetric with respect to the same axis then there exists the circle inscribed to given quadrilateral. If the symmetry axis are distinct then the bissectors of two angles formed by the opposite sideline are perpendicular and so the quadrilateral is cyclic.

10. This is the reformulation of previous problem for the quadrilateral formed by the lines AX, BX, AY, BY.

11. By the theorems 1, 12 there exists a parabola with focus F touching the sidelines of the triangle and two perpendicular lines passing through its orthocenter H. The midpoints of stiked segments are the circumcenters of respective rectangular triangles, so they lie in medial perpendicular to HF.

12. If the triangle ABC is rectangular and isosceles ($AC = BC$) and C^* is the infinite point perpendicular to AB then the locus of points P is the circumcenter of ABC. Using respective projective transformation we obtain that in general case the locus of P is the circumconic.

Now as this conic pass through the orthocenter it is the equilateral hyperbola and by theorem 15 all circles $A'B'C'$ pass through its center.

13. The orthocenter, the incenter the Gergonne point and the Nagel point are isogonally conjugated to the circumcenter, the incenter and two homothety centers of the incircle and the circumcircle respectively. So the isogonal images of four points lie on the line passing through O. By the theorem 9 these points lie on the equilateral circumhiperbola and we can use the assertion of previous problem.

14. The pedal circles of two isogonal points coincide. So the assertion of problem follows from the theorems 9, 15.

15. The similarity $\mathcal{F}$ transforming one of given triangles to the other is the composition of reflection in some line l passing through common orthocenter H and homothety with center H. Let Γ be an equilateral circumhyperbola of given triangle ABC such that one of its asymptotes is parallel to l. Similarly Γ' is the equilateral hyperbola of second triangle $A'B'C'$ with the same directions of asymptotes. Γ and Γ' have three real common points: H, and two infinite points. So their fourth common point P is also real. $\mathcal{F}$ define projective tranformation from Γ to Γ' such that three common points of hyperbolaes leave fixed. The single transformation with this property is the projection from fourth common point P. So P is the perspective center of given triangles.