Three parabolas
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We will examine the properties of the next configuration: a triangle and three parabolas, each passing through two vertices and touching in these vertices the respective sidelines.

1 Necessary definitions and theorems

Definition 1. *A parabola* is the locus of points with equal distances from a fixed point F and a fixed line l. F is *the focus* and l is *the directrix* of the parabola.

Definiton 2. Given triangle ABC and point P. Then the reflections of lines AP, BP, CP in respective bisectors concur in point P'. This point is called *isogonally conjugate* to P. Point L, isogonally conjugate to the centroid M, is called *the Lemoine point.*

Definition 3. The projections of point P to the sidelines of ABC form the triangle called *the pedal triangle* of P wrt ABC. The circumcircle of the pedal triangle is called *the pedal circle* of P.

Definition 4. The triangles ABC and $A'B'C'$ are called *perspective* if the lines AA', BB' and CC' concur. The common point of these lines is called *the perspective center.*

The Desargues theorem. Two triangles are perspective iff the common points of their respective sidelines are collinear. The line passing through these points is called *the perspective axis.*

Definition 5. Triangles ABC and $A'B'C'$ are called *orthological* if the perpendiculars from A to $B'C'$, from B to $A'C'$ and from C to $A'B'$ concur. The common point of these perpendiculars is called *the orthology center.* This property is reciprocal, so the orthological triangles have two orthology centers, distinct in general.

The centroid M, the circumcenter O and the orthocenter H of an arbitrary triangle lie on the line called *the Euler line.* The midpoints of the sides and the feet of the altitudes lie on the circle called *the Euler circle.*

Definition 6. For an arbitrary triangle ABC, there exist two points Q_1, Q_2, such that $\angle Q_1AB = \angle Q_1BC = \angle Q_1CA$ and $\angle Q_2BA = \angle Q_2CB = \angle Q_2AC$. These points are called *the Brocard points* of ABC. The angle $\phi = \angle Q_1AB = \angle Q_2BA$ is called *the Brocard angle.*

For any non-regular triangle there exist two points with regular pedal triangle. These points are inverse wrt the circumcircle of the triangle. The point lying inside the circumcircle is called *the first Apollonius point*, another point is called *the second Apollonius point.* Two points isogonally conjugate to the Apollonius points are called *the first and the second Torricelli points.*

2 Introductory problems

1. Prove *the optic property* of the parabola: the tangent to a parabola in a point X forms equal angles with the line FX and the axis of parabola.

2. Prove that the reflection of the focus in the tangent to the parabola in point X coincides with the projection of X to the directrix.

3. Let P be the common point of tangents to parabola in points X and Y. Prove that P is the circumcenter of triangle $FX'Y'$, where X', Y' are the projections of X and Y to the directrix.

4. Prove that in notation of the previous problem, the median of PXY is parallel to the axis of parabola.

5. Prove that in notation of problem 3, the bisector of angle P forms equal angles with the line PF and the axis of the parabola.

6. Prove that two isogonally conjugate points have common pedal circle.

   Given triangle ABC. Denote by Π_c the parabola touching the lines AC и BC in points A and B. Furthermore denote by F_c the focus of Π_c. The parabolas Π_a, Π_b and their focuses F_a, F_b are defined similarly.

7. Prove that Π_a and Π_b have only one common point C', different from C.

8. Define points A', B' similarly to C'. Prove that triangles ABC and $A'B'C'$ are perspective.

9. Prove that the triangles ABC and $F_aF_bF_c$ are perspective.

10. Prove that the triangle formed by the directrices of Π_a, Π_b, Π_c is perspective to ABC.

3 Basic problems

11. Prove that F_cC is the bisector of angle AF_cB.

12. Prove that A, B, F_c and the circumcenter O of ABC are cocyclic.

13. Prove that the medial line parallel to AB touches Π_c.

14. Prove that F_a, F_b, F_c, O, L are cocyclic.

15. Prove that the directrix of Π_c, the median from C and the Euler circle concur.

16. Prove that the centroids of ABC and of the triangle formed by three directrices coincide.

17. Prove that ABC and the directrix triangle are orthological and their orthology centers coincide.

18. Prove that the Euler line of ABC passes through the Lemoine point of the triangle formed by three directrices.

19. Prove that the Euler line of the directrix triangle passes through L.

20. Let AF_a, BF_b, CF_c intersect the respective sidelines of ABC in the points P_a, P_b, P_c. Prove that triangle $P_aP_bP_c$ is perspective to the directrix triangle.

21. Prove that the pairwise perspective centers of ABC, $P_aP_bP_c$ and of the directrix triangle are collinear. The respective line passes through the midpoint of the segment between the centroids of ABC and $P_aP_bP_c$.

22. Prove that the directrix triangle is perspective to the orthotriangle of ABC.

23. Prove that the Brocard angles of ABC and the directrix triangle are equal.

4 Additional problems

24. Let T be some Torricelli point of ABC and T_d the respective Torricelli point of the directrix triangle. Prove that the lines TA, TB, TC are parallel to the lines joining T_d with respective vertices of the directrix triangle.

25. Prove that TT_d and the line joining the respective Apollonius points pass through the centroid of ABC.

26. Find $\frac{S}{S'}$, where S' is the area of the curvilinear "parabolic"triangle $A'B'C'$, and S is the area of ABC.

27. Let N_1, N_2 be the perspective centers of the directrix triangle and the triangles ABC, $P_aP_bP_c$ respectively. Prove that the line N_1N_2 bisects the segment between the centroids of ABC and $P_aP_bP_c$.

28. Let F_a', F_b', F_c' be isogonally conjugate to F_a, F_b, F_c; H is the orthocenter, M the centroid of ABC. Prove that

    a) F_a', F_b', F_c' lie on the circle with diameter HM.

    b) The distances from F_a and F_a' to the center of the Euler circle of ABC are equal.

    The point of parabola Π_c, nearest to point C, will be called the *projection* of point C to the parabola and will be denoted by C^*.

29. Prove that line CC^* is perpendicular to parabola Π_c.

30. Prove that $\angle AC^*C = \angle BC^*C$.

31. Denote by C_1 and C_2 the common points of the tangent to the parabola at point C^* with CA and CB respectively. Denote by C^{**} the common point of C_1C_2 and the median of triangle ABC drawn from vertex C. Prove that $C_1C^* = C^{**}C_2$.

    Define points A^* and B^* similarly to C^*. r liner

32. (**Conjecture.**) Lines AA^*, BB^* and CC^* concur.

5 Related problems

Consider the set of lines bissecting the perimeter of triangle ABC.

33. Prove that these lines envelop three parabolas, each touching two sidelines of ABC in the same points that the respective excircle.

    Note by Π_A the parabola touching AB and AC. Define Π_B and Π_C similarly.

34. Prove that the circumcircle of ABC is the orthology center of ABC and the triangle formed by the directrix of Π_A, Π_B and Π_C.

35. Prove that the orthocenter of ABC lies on the Euler line of triangle formed by the directrix.

36. Prove that the incenter of ABC and the circumcenter of directrix triangle are symmetric wrt the center of Euler circle of ABC.

37. Let T be some Torricelli point of triangle ABC, T_d be the respective point of directrix triangle. Prove that the lines, joining T and T_d with respective vertex are parallel.

6 Related problems-2

Consider points A_c and A_b on rays AB and AC such that $CA = CA_c$ and $BA = BA_b$. Let parabola Π'_a touche AC and AB in points A_b and A_c. Parabolas Π'_b and Π'_c are defined similarly.

38. Prove that the focus of Π'_a, Π'_b and Π'_c coincide with F'_a, F'_b and F'_c.

39. Prove that the triangle formed by directrix of Π'_a, Π'_b and Π'_c is perspective to ABC.

40. Prove that the directrix triangle of Π'_a, Π'_b and Π'_c is ortologic to ABC and their othology centers coincide.

41. Prove that the directrix triangle of Π'_a, Π'_b and Π'_c is perspective to the orthotriangle of ABC.

42. Let O' and M' be the circumcenter and the centroid of directrix triangle. Prove that the common point of lines $O'M$ and $M'O$ coincide with the Lemoine point of ABC.