Weighings using broken balances

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1 Introduction

Basic notions

As in the usual weighing problems, we have a set of coins indistinguishable by sight, though one of them is false. The weight of a false coin is a bit smaller than the weight of a genuine one, ans we want to find this false coin. In different problems we will use the different types of testers.

A *detector* is a tester, which at one turn (or *testing*) tells whether a given subset of coins contains a false one or not. Thus, for any subdivision of the total set into two subsets, a detector shows which one of them contains a false coin.

A bi-scaled balance (or simply *balance*) is a tester comparing the weights of two subsets. Thus, for two subsets with equal numbers of coins, the balance compares the numbers of false coins in these subsets. In our problem, the balance shows which of three subsets in a subdivision contains a false coin (two of three subsets should have the same cardinality).

The zest of this problem set is that some testers can report a wrong information. Actually, such a tester does not behave as a "liar": it does **not** necessarily tell the wrong messages each time. It is simply broken, so its responses may appear wrong and correct from time to time; they have no relation to the reality, and we can consider this balance as a generator of random answers. So, we have several testers, and we know only the number of broken testers among them; but we do not know, which ones are broken. (Note that at each testing, only one tester is used!)

Let us introduce the notation. We denote by $\mathrm{D}_{x,y}(z)$ the minimal number of testings by detectors which are necessary to find one false coin from n using x detectors with y broken ones among them. (We should be able to specify the false coin for **each** possible sequence of detector responses.) Analogously, by $\mathrm{B}_{x,y}(z)$ we denote the same number for the testing system consisting of x balances with y broken ones among them. The systems of x detectors (balances) with y broken ones will be referred to as $x\mathrm{d}[y]$ ($x\mathrm{b}[y]$).

Throughout the first sections, we will write down two formulations of the problem: one in an usual language, and another in our notation. This is to get all the readers acquainted with the notation.

2 Introductory problems: some particular cases

What are we able to achieve?

2.1. Using three balances with one broken among them, we can find a false coin of 3 ones in 3 weighings. (Using the notation: $\mathrm{B}_{3,1}(3) \le 3$.)

2.2. ***a)*** Using three detectors with one broken among them, we can find a false coin of 8 ones in 6 detections. (Using the notation: $\mathrm{D}_{3,1}(8) \le 6$.)

b) Using three balances with one broken among them, we can find a false coin of 9 ones in 4 weighings. (Using the notation: $\mathrm{B}_{3,1}(9) \le 4$.)

2.3. ***a)*** Using three detectors with one broken among them, we can find a false coin of 32 ones in 9 detections. (Using the notation: $\mathrm{D}_{3,1}(32) \le 9$.)

b) Using three balances with one broken among them, we can find a false coin of 81 ones in 7 weighings. (Using the notation: $\mathrm{B}_{3,1}(81) \le 7$.)

What are we unable?

2.4. If we have two coins with one of them being false, it is not possible to find the false one in 2 testings using any number of testers with at least one broken among them. (Using the notation: $\mathrm{D}_{x,1}(2) \geq 3$, $\mathrm{B}_{x,1}(2) \geq 3$.)

2.5. ***a)*** It is not possible to find a false coin from 2^k ones in k detections by any number of detectors with one broken among them. (Using the notation: $\mathrm{D}_{x,1}(2^k) > k$).

b) It is not possible to find a false coin from 3^k ones in k weighings by any number of balances with one broken among them. (Using the notation: $\mathrm{B}_{x,1}(3^k) > k$).

2.6. ***a)*** It is not possible to find a false coin from 2^k ones in $k+1$ detections by any number of detectors with one broken among them. (Using the notation: $\mathrm{D}_{x,1}(2^k) > k+1$).

b) It is not possible to find a false coin from 3^k ones in $k+1$ weighings by any number of balances with one broken among them. (Using the notation: $\mathrm{B}_{x,1}(3^k) > k+1$).

c) It is not possible to find a false coin from $n > 3^6$ ones in 11 weighings by any number of balances with **two** broken ones among them. (Using the notation: $\mathrm{B}_{x,2}(n) > 11$, if $n > 3^6$).

3 Rough but serial results

Upper bounds

In this section, k is always a positive integer.

3.1. ***a)*** For each k find the minimal value of K such that $\mathrm{D}_{K,k}(n) < \infty$ for every n (in other words, find the least number of detectors with k broken among them such that it is possible to find the false coin using them).

b) For each k find the minimal value of K such that $\mathrm{B}_{K,k}(n) < \infty$ for every n.

3.2. ***a)*** Prove that $\mathrm{D}_{3,1}(2^k) \leq 2k+1$.

b) Prove that $\mathrm{B}_{3,1}(3^k) \leq 2k+1$.

It follows from the previous problem that, using some balances with one broken, one can find a false coin from n in approximately $2\log_2 n$ detections, or in approximately $2\log_3 n$ weighings. But this estimate is not sharp. The aim of this section is to find the correct constant in the estimates of the form $\mathrm{D}_{x,1}(n) \lesssim c\log_2 n$ and $\mathrm{B}_{x,1}(n) \lesssim c\log_3 n$.

3.3. ***a)*** Prove that $\mathrm{B}_{3,1}(3^{2k}) \leq 3k+1$.

b) Prove that $\mathrm{D}_{3,1}(2^{2k}) \leq 3k+2$.

The previous problem shows that c can be made less than 2. Our next aim is to prove that $c = 1$. It is easier to make this using more than 3 testers.

We write $f(k) = o(k)$ for the function growing slower than k, that is, $f(k)/k \to 0$ $(k \to \infty)$. For instance, $\log_2 k = o(k)$ and $\sqrt{k} = o(k)$.

3.4. ***a)*** Prove that having an infinite number of detectors with one broken, one can find a false coin from 2^k in $k + o(k)$ weighings; that is, $\mathrm{D}_{\infty,1}(2^k) = k + o(k)$.

b) Prove that having an infinite number of balances with one broken, one can find a false coin from 3^k in $k + o(k)$ weighings; that is, $\mathrm{B}_{\infty,1}(2^k) = k + o(k)$.

3.5. ***a)*** Prove that there exists x such that $\mathrm{D}_{x,1}(2^k) = k + o(k)$.

b) Prove that there exists x such that $\mathrm{B}_{x,1}(3^k) = k + o(k)$.

3.6. ***a)*** Prove that $\mathrm{D}_{3,1}(2^{k(k+1)}) \leq (k+1)^2$ for every $k \geq 5$.

b) Prove that $\mathrm{B}_{3,1}(3^{k(k+1)}) \leq (k+1)^2$ for every $k \geq 2$.

Lower bounds

3.7. ***a)*** Prove that $\mathrm{D}_{x,1}(n) \leq \mathrm{D}_{x,1}(2^k)$ if $n < 2^k$.

b) Prove that $\mathrm{B}_{x,1}(n) \leq \mathrm{B}_{x,1}(3^k)$ if $n < 3^k$.

3.8. ***a)*** Prove that $\mathrm{D}_{x,1}(n) \leq \mathrm{D}_{x,1}(N)$ if $n < N$.

b) Prove that $\mathrm{B}_{x,1}(n) \leq \mathrm{B}_{x,1}(N)$ if $n < N$.

3.9. *a)* Suppose that $\mathrm{D}_{x,1}(n) = d$; prove that $\dfrac{2^d}{d+1} \geq n$. (This bound does not depend on x!)

b) Suppose that $\mathrm{B}_{x,1}(n) = d$; prove that $\dfrac{3^d}{2d+1} \geq n$.

4 Sharp results

The aim of this section is to bring upper and lower bounds together. So it begins with some particular cases again.

4.1. *a)* Prove that $\mathrm{D}_{4,1}(2^4) = 7$.

b) Prove that $\mathrm{B}_{4,1}(3^6) = 9$.

4.2. *a)* Find the maximal number of coins n such that $\mathrm{D}_{4,1}(n) \leq 15$.

b) Find the maximal number of coins n such that $\mathrm{B}_{4,1}(n) \leq 13$.

c) Find the maximal number of coins n such that $\mathrm{B}_{4,1}(n) \leq 40$.

4.3. Find some value of n such that $\mathrm{B}_{4,1}(n) < \mathrm{B}_{3,1}(n)$.

4.4. *a)* Prove that $\mathrm{D}_{4,1}(3^k) = k + \log_2 k + c_{\mathrm{d}k}$, where the sequence $c_{\mathrm{d}k}$ is bounded.

b) Prove that $\mathrm{B}_{4,1}(3^k) = k + \log_3 k + c_{\mathrm{b}k}$, where the sequence $c_{\mathrm{b}k}$ is bounded.

c) Try to find a better upper bound for these sequences.

4.5. *a)* Prove that $\mathrm{D}_{x,1}(n) = \mathrm{D}_{4,1}(n)$ for every n and $x > 4$.

b) Prove that $\mathrm{B}_{x,1}(n) = \mathrm{B}_{4,1}(n)$ for every n and $x > 4$.

Fix some value of s. If there exists t such that $\mathrm{D}_{t,s}(n) = \mathrm{D}_{x,s}(n)$ ($\mathrm{B}_{t,s}(n) = \mathrm{B}_{x,s}(n)$) for all $x > t$, then we say that t is an *ideal* number of testers (for he given number s of broken ones). In other words, it is senseless to increase the number of testers. The previous problem states that 4 testers is an ideal number for 1 broken tester.

4.6. *a)* Find whether the estimate of the same form as in problem 4.4 is valid for $\mathrm{D}_{3,1}(n)$.

b) The same question for $\mathrm{B}_{3,1}(n)$.