Weighings using broken balances. Solutions

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2 Introductory problems: some particular cases

What are we able to achieve?

For convenience, we will denote a false coin as FC.

Note that in the situation with only one broken tester, if the readings of two testers on identical questions coincide, then this reading is true since one of the testers is operable.

2.1. Statement. *Using three balances with one broken among them, we can find a false coin of 3 ones in 3 weighings. (Using the notation: $B_{3,1}(3) \leqslant 3$.)*

Solution. Compare two coins consecutively on all three balances. At least two results will coincide — these ones will be correct. From this result, one can easily find a FC.

2.2. ***a*) Statement.** *Using three detectors with one broken among them, we can find a false coin of 8 ones in 6 detections. (Using the notation: $D_{3,1}(8) \leqslant 6$.)*

Solution. Split the coins into 4 groups with 2 coins in each. Apply the first detector to groups 1 and 2, and the second one to groups 1 and 3. Without loss of generality, we may assume that both answers are affirmative (why?). Then apply the third detector to groups 2 and 3. Two cases may occur.

1. If the answer is negative, then FC is in group 1 (otherwise two balances made wrong statements). Then we check one coin from this group by all three detectors; these results show which coin from group 1 is false.

2. Suppose that the third answer is affirmative. Then one of detectors lied (note that if we had the fourth detector, then it would be operable, and the remaining would be easy; nevertheless, we can also proceed without it!).

Check groups 1 and 2 — now by the second detector. If the result is affirmative (so it coincides with the result of the first detector), then it is true, and FC is in group 1 or 2. Moreover, the first balance told the truth on first use, hence either second or third balance lied. Otherwise, if the result of the third detection is negative, then either first or second balance is broken; hence the third one is operable, and FC is in group 2 or 3.

So, in each case we get one operable balance and 4 candidates to be FC. It is now easy to find FC in 2 detections.

***b*) Statement.** *Using three balances with one broken among them, we can find a false coin of 9 ones in 4 weighings. (Using the notation: $B_{3,1}(9) \leqslant 4$.)*

Solution. For the convenience, we enumerate the coins by numbers from 0 to 8, and write these numbers in a ternary notation (so, each coin corresponds to an ordered pair of digits from 0 to 2). In the first weighing, we apply the first balance to the groups constructed by the leftmost digit: the left hand contains the coins with 0 at this digit, and the right hand contains those with 1. Analogously, the second weighing (on the second balance) compares the coins with 0 in the last digit with those with 1. Without loss of generality, we may assume that both weighings told that a false coin has 0 in a corresponding digit. Then 4 coins having no zeroes in a ternary notation are surely genuine.

Now we split all the coins into three parts in a following way: put coin 00 into one group, coins 01 and 02 into the second one, and 10 and 20 into the third. Complete each group by genuine coins so that each group would contain 3 coins. Compare two of these groups by the third balance.

This weighing claims that FC is in one of these groups. If this is a group with 00, then this is true (otherwise two balances lie), and we do not need the fourth weighing. If this is a group with 01 and 02, then this claim contradicts the readings of the second balance. Hence, the first balance is operable, so the false coin has 0 as its first digit, and there are three candidates for FC. Thus, we can easily find FC in one weighing. If the false coin in a group with 10 and 20, we proceed analogously.

2.3. ***a*) Statement.** *Using three detectors with one broken among them, we can find a false coin of 32 ones in 9 detections. (Using the notation: $D_{3,1}(32) \leqslant 9$.)*

Solution. Split all the coins into 4 groups with 8 coins in each, and apply the same algorithm as in 2.2a). In case 1, after 3 detections we get 8 candidates for FC; applying problem 2.2a) again, we make the desired. In case 2, in 4 detections we find two groups with FC in one of them, and we find one operable detector. So in 4 other detections we find FC; in this case we do not need a 9th detection.

***b*) Statement.** *Using three balances with one broken among them, we can find a false coin of 81 ones in 7 weighings. (Using the notation: $B_{3,1}(81) \leqslant 7$.)*

Solution. Analogously to problem 2.2b), we identify the coins with the sequences of 4 digits, each being from 0 to 2. Again, we take the first weighing on first balance according to the first digit of a coin, and the second weighing on the second balance by the second digit. Again, without loss of generality we may assume that zero digits are suspicious by opinions of the balances, and then the coins with nonzero first two digits are genuine.

Now we again split coins into three groups: first one containing 9 coins beginning with 00, the second containing 18 coins beginning with 01 and 02, and the third one containing the coins beginning with 10 or 20. we can distribute the genuine coins so that each group would contain 27 coins; then we compare two groups on the third balance.

If the result claims that FM is in the group beginning with 00, then this claim is true, and we are left to work out 9 coins in 4 weighings; this is possible by problem 2.2b).

Otherwise, suppose that the result claims that the false coin begins either with 01 or with 02 (the case with 10 and 20 is analogous). Then the readings of the third balance contradict those of the second balance; hence the first one is surely operable, and (by the first weighing) the false coin should begin with 0. Thus we get 27 coins and one operable balance; then we are able to find FC in 3 weighings (in this case we do not need the 7th weighing).

What are we unable?

2.4. Statement. *If we have two coins with one of them being false, it is not possible to find the false one in 2 testings using any number of testers with at least one broken among them. (Using the notation: $D_{x,1}(2) \geqslant 3$, $B_{x,1}(2) \geqslant 3$.)*

Solution. If both weighings are made on one tester, then it can appear to be broken, and we get no information at all. Otherwise, we cannot find FC if the readings of two balances contradict each other.

2.5. a) Statement. *It is not possible to find a false coin from 2^k ones in k detections by any number of detectors with one broken among them. (Using the notation: $D_{x,1}(2^k) > k$).*

Solution. Suppose this is possible.

Suppose that some tests are already taken. Consider all the coins which can appear to be false. For each of them, there exists a result of the next test which allows this coin to remain false. Hence the number of such coins divides by not more than 2 for some result of the next test.

Note that this number did not change on the first test, since this test might use the broken tester. Hence after d steps (with an appropriate choice of test results) we get not less than $2^k/2^d$ possible FCs. Hence, if we find FC, then $2^k/2^d \leqslant 1 \iff d \geqslant k$, QED.

b) Statement. *It is not possible to find a false coin from 3^k ones in k weighings by any number of balances with one broken among them. (Using the notation: $B_{x,1}(3^k) > k$).*

Solution. Analogously, but the number of candidates divides by 3 instead of 2.

2.6. a) Statement. *It is not possible to find a false coin from 2^k ones in $k+1$ detections by any number of detectors with one broken among them. (Using the notation: $D_{x,1}(2^k) > k+1$).*

Solution. We say that a testing *refuses* to a coin, if it claimed that this coin is not in a group with a false one. In other words, if a testing refused to a coin, then either the coin is genuine or the tester is broken.

By the same reasons as in a previous problem, if one can find FC in $k+1$ testing, then after the second testing we get not more than 2^{k-1} candidates to be false. On the other hand, each coin, to which at least one of first two testings did not refuse, is such a candidate (if another testing was made on a broken detector). Let us estimate the number of such coins.

The first testing (for one of the results) refuses to not more than $\frac{1}{2}$ of coins; hence it *does not refuse* to a half at least. The second testing — for one of the results — does not refuse to some coin such that the first one refused to it. Then, the number of candidates is more than a half on a total number of coins; hence the desired is not possible.

b) Statement. *It is not possible to find a false coin from 3^k ones in $k+1$ weighings by any number of balances with one broken among them. (Using the notation: $B_{x,1}(3^k) > k+1$).*

Solution. Analogously: we get that the first weighing does not refuse to at least a third of the total number of coins, and the second one does not refuse to at least one more coin. Hence, after these weighings the number of candidates is greater than a third of the total number of coins.

c) Statement. *It is not possible to find a false coin from $n > 3^6$ ones in 11 weighings by any number of balances with* **two** *broken ones among them. (Using the notation: $B_{x,2}(n) > 11$, if $n > 3^6$).*

Solution. The solution of this problem is in the end of Section 3.

3 Rough but serial results

Upper bounds

3.1. Statement. a) *For each k find the minimal value of K such that $D_{K,k}(n) < \infty$ for every n (in other words, find the least number of detectors with k broken among them such that it is possible to find the false coin using them).*

b) *For each k find the minimal value of K such that $B_{K,k}(n) < \infty$ for every n.*

Solution.

The solution does not depend on the type of testers.

Answer. $K = 2k+1$.

Suppose the number of testers is $\leqslant 2k$, and the coin A is false. Let all broken testers work so as the false coin is B. In this case we cannot determine whether all broken testers are broken (and the false coin is A) or we have the opposite case (and the false coin is B). If the number of testers is $2k+1$ then each coin (or, in the case of balances, each pair of coins) can be tested by all testers; $k+1$ results will coincide, and this will be the valid result.

3.2. a) Statement. *Prove that $D_{3,1}(2^k) \leqslant 2k+1$.*

Solution. We use induction on k. For $k=1$ we test the first coin by all three detectors; two coincident answers are valid.

Suppose the assertion is proved for $k = t$. Consider 2^{t+1} coins and test first 2^t of them by first two detectors. If the results coincide then they are valid, so we reduce the number of coins to 2^t; by the induction hypothesis, we obtain our assertion.

In the case of distinct results, one of two first detectors is broken; then the third (apparently operable) detector can find FC in $t+1$ detections; in this case we have used not more than $t + 3 \leqslant 2(t + 1) + 1$ detections.

b) **Statement.** *Prove that* $B_{3,1}(3^k) \leqslant 2k + 1$.

Solution. Quite similarly.

3.3. ***a)*** **Statement.** *Prove that* $B_{3,1}(3^{2k}) \leqslant 3k + 1$.

Solution. Again by induction; the base case $k = 1$ is Problem 2.2b). The induction step is quite similar to the solution of 2.3b).

b) **Statement.** *Prove that* $D_{3,1}(2^{2k}) \leqslant 3k + 2$.

Solution. Similarly, by induction; the induction step is similar to the solution of 2.3a). The base case is proved by a similar argument; namely, we either find FC in 3 detections, or in 4 detections we find a pair of coins including the false one, and an operable detector. Then the remaining testing enables to find FC.

3.4. ***a)*** **Statement.** *Prove that having an infinite number of detectors with one broken, one can find a false coin from* 2^k *in* $k + o(k)$ *weighings; that is,* $D_{\infty,1}(2^k) = k + o(k)$.

Solution. Enumerate all coins by k-digit binary numbers (from $0 \dots 0$ to $1 \dots 1$). In the first k testings we apply first k detectors, each to the corresponding position. Without loss of generality, we may assume that each detector responses that the false coin has 0 in the corresponding position. Hence the number corresponding to FC contains not more than one 1.

Apply $(k + 1)$th detector to the coin $0 \dots 0$. If the response is that it is false then this is valid (otherwise two detectors lie!). In the opposite case, some detector among first $k + 1$ ones has lied, and we have $k + 1$ coins possibly false. Among them, we can find FC by $(k + 2)$th (operable!) detector in $\leqslant \log_2(k + 1) + 1$ testings. In all, we have used not more than $k + 1 + \log_2(k + 1) + 1 = k + o(k)$ testings.

b) **Statement.** *Prove that having an infinite number of balances with one broken, one can find a false coin from* 3^k *in* $k + o(k)$ *weighings; that is,* $B_{\infty,1}(2^k) = k + o(k)$.

Solution. Quite similarly.

3.5. ***a)*** **Statement.** *Prove that there exists* x *such that* $D_{x,1}(2^k) = k + o(k)$.

b) *Prove that there exists* x *such that* $B_{x,1}(3^k) = k + o(k)$.

Solution. Follows from the next problem.

3.6. ***b)*** **Statement.** *Prove that* $B_{3,1}(3^{k(k+1)}) \leqslant (k + 1)^2$ *for every* $k \geqslant 2$.

Solution. We will use induction on k. The base case $k = 1$ was proved in 2.2.b.

As above, enumerate coins in the ternary notation. In the first k weighing, we examine first k position by the first balance, and in the next k weighings we examine the next k positions by the second balance. Without loss of generality, we may assume that according to all weighings, the false coin has 0 in the corresponding positions. Now if FC has not 0 both in the first and in the second group of positions, then both balances have lied. Hence all such coins are genuine (we call them *standards*).

For the $(2k+1)$th weighing, we split all non-standards as follows. The first group includes coins having only zeros in the first $2k$ positions; the second group includes coins having only zeros in the first k positions but not only zeros in the next k positions; and the third group consists of coins which have not only zeros in the first k positions but only zeros in the next k positions. Clearly the second and the third group include equal number of coins; apply the third balance to them. If the weighing indicates the first group then the results of all weighings are valid (otherwise at least two balances lied); hence it remains to determine $k(k - 1)$ positions in k^2 weighings which is possible by the induction hypothesis. If k^2th weighing indicates the second group (similarly for the third group) then the indicating of the second and of the third balance contradict each other. Hence the first balance is definitely operable. Then we already know the first k positions, and in the remaining k^2 testing, we can determine the remaining positions by the operable balance.

a) **Statement.** *Prove that* $D_{3,1}(2^{k(k+1)}) \leqslant (k + 1)^2$ *for every* $k \geqslant 5$.

Solution. An estimate analogous to part b) would be weaker, namely, $\mathrm{D}_{3,1}(2^{k(k-1)/2-1}) \leqslant k(k + 1)/2 - 1$. Unfortunately, it seems that this problem does not have a solution much easier than 4.4.

Lower bounds

3.7. ***a)*** **Statement.** *Prove that* $D_{x,1}(n) \leqslant D_{x,1}(2^k)$ *if* $n < 2^k$.

Solution. See the solution of 3.8a).

b) **Statement.** *Prove that* $B_{x,1}(n) \leqslant B_{x,1}(3^k)$ *if* $n < 3^k$.

Solution. The authors know no simple proof of this fact.

3.8. ***a)*** **Statement.** *Prove that* $D_{x,1}(n) \leqslant D_{x,1}(N)$ *if* $n < N$.

Solution. Suppose we have an algorithm which finds the false coin among N coins. Consider n coins and add $N - n$ dummy ones to them. Apply the same algorithm to them; and if we have to put a dummy coin to the detector, we put nothing. Clearly we will find FC by the same number of testings.

b) **Statement.** *Prove that* $B_{x,1}(n) \leqslant B_{x,1}(N)$ *if* $n < N$.

Solution. The authors know no simple proof of this fact.

3.9. We will present two solutions of the problem. Both are valid for both parts of the problem with minor changes; we will present one of them for the first part, and another one for the second part.

a) **Statement.** *Suppose that* $D_{x,1}(n) = d$*; prove that* $\dfrac{2^d}{d+1} \geqslant n$*. (This bound does not depend on* x*!)*

Solution. Consider an algorithm which finds FC in d moves. Suppose we have executed such an algorithm; write down the results of detections in a line. We obtain a sequence of d symbols "Y" and "N" (meaning "yes" and "no"). This sequence enables us to reconstruct the detections performed (since the first i results determine uniquely the detection performed at the $(i+1)$-th step), hence each of these sequences uniquely determines an FC.

Now let us find out, how many such sequences correspond to an arbitrary FC. Their number is not less than $d+1$; in fact, at any rate we have the following sequences: 0) with all answers valid; 1) with just the first answer wrong; 2) with just the second answer wrong; ...; k) with just the kth answer wrong. Obviously sequence 0) differs from all the others; and sequences i) and j) (for $i < j$) differ at least in the ith position.

Thus the number of sequences is not less than $n(d+1)$; on the other hand, the possible number of sequences does not exceed 2^d. Hence $n(d+1) \leqslant 2^d$ as required.

Remark. This proof shows how should we arrange the algorithm if we wish to approach the above estimate. The latter will be precise if a) (almost) all sequences are possible and b) (!!!) if any tester made a wrong statement then we can recognize this before its next use. Actually, if this is not done then the number of sequences corresponding to the same FC increases.

b) **Statement.** *Suppose that* $B_{x,1}(n) = d$*; prove that* $\dfrac{3^d}{2d+1} \geqslant n$*.*

Solution. We use the following notation. For each weighing, we split the coins in three groups, and the next balance refuses to coins of two groups. Suppose some weighings have been already performed. If some coin was **always** in the "false" group when using ith balance, then we shall call it *false relative to* this balance.

After several steps of the algorithm, all coins may be split into the following groups. 1) Coins which are false relative to all balances; we will call these coins *suspected.* 2) Coins which are false relative to all balances except the ith one; we will call them *refusers of the* i*th type.* Clearly these coins can occur to be false only in the case when the ith balance is broken. 3) Coins tested as genuine by at least two balances; then they are in fact genuine, and we will call them *standard.*

Suppose we have an algorithm which allows to find the false coin among n coins in d steps. We may assume that it finishes its work always in just d steps (if before then we carry out several arbitrary weighings).

We will introduce the notion of the *significance* of a coin at some moment of performing the algorithm. If we have m steps since that till the end of the algorithm (that is, $d-m$ steps are over), we will say that the significance of any suspected coin equals $2m+1$, and the significance of any refuser is 1 (the significance of a standard equals 0). The significance of a situation equals the sum of significances of all coins.

Note that if in a situation of significance x we perform a weighing (for instance, by the first balance) then the sum of significances of three possible situations in which the weighing can result is not less than x. It suffices to prove that the sum of possible significances of each coin after weighing is not less than the initial significance. A suspected coin of significance $2m+1$ remains suspected for a single result (with significance $2(m-1)+1$)), and two other results make it a refuser of significance 1. A "not-first-type" refuser remains a refuser for one result, and for two others it becomes a standard. And for refusers of the first type (and only for them!) the sum of significanes after weighing exceeds their initial significance (1 turns to 3).

Thus, if the initial significance of the situation was x then some of the results makes it not less $x/3$. Suppose that we have such a result in each weighing. The initial significance was equal to $(2d+1)n$ (all coins were suspected), and in the end it must be 1 (because after d steps the significance of any coin equals 1). Hence $(2d+1)n \leqslant 3^d$ as required.

Remark. This proof also is rather informative as regards the form of the optimal algorithm. Namely, each step must make the significance three times smaller.

Now let us realize what can be an obstacle to make the significance three times smaller at each step? First, if we use a balance of a type containing a refuser then the significance of these refusers does not decrease thrice. Hence we have to arrange the process so that this situation does not occur. Second, for each weighing the significances of situations arising for three possible results have to be roughly equal.

2.6. ***c)*** **Statement.** *It is not possible to find a false coin from* $n > 3^6$ *ones in* 11 *weighings by any number of balances with* **two** *broken ones among them. (Using the notation:* $B_{x,2}(n) > 11$*, if* $n > 3^6$*).*

Solution. Consider an arbitrary algorithm which enables to find FC among n coins in 11 steps. Suppose we have carried it out; write down the results of detections in a line. We obtain a sequence of 11 symbols "<", "=" and ">". By the same reasons as in 3.9a), each of these sequences uniquely determines an FC.

Now we shall find the number of sequences corresponding to the same FC. This number is not less than $1 + 2 \cdot 11 + 4 \cdot 55 = 243$; indeed, we have at least the sequences of the following types: 0) all answers are valid; 1) just one answer is wrong (the number of such sequences is 22: there are 11 ways to choose the position for the wrong answer, and each of them may contain one of two possible wrong answers); 2) just 2 answers are wrong (the number of such sequences is $4 \cdot 55$, since there are 55 pairs of positions and 4 pairs of wrong answers). Clearly all the above sequences are distinct.

Thus the number of sequences is not less than $243n$; on the other hand, the number of all possible sequences does not exceed 3^{11}. Hence $n \cdot 243 \leqslant 3^{11}$, and so $n \leqslant 3^6$.

4 Sharp results

4.1. Statement. ***a)*** *Prove that $D_{4,1}(2^4) = 7$.*

b) *Prove that $B_{4,1}(3^6) = 9$.*

Solution. See the solution 4.4. Nevertheless, these problems can be solved without use of such general methods.

4.2. Statement. ***a)*** *Find the maximal number of coins n such that $D_{4,1}(n) \leqslant 15$.*

b) *Find the maximal number of coins n such that $B_{4,1}(n) \leqslant 13$.*

c) *Find the maximal number of coins n such that $B_{4,1}(n) \leqslant 40$.*

Solution. Answers. a) 2^{11}; b) 3^{10}; c) 3^{36}.

The upper bounds follow from 3.9. Their attainability follows from the solution of 4.4. Nevertheless it is possible to obtain the corresponding algorithms by direct methods.

4.3. Statement. *Find some value of n such that $B_{4,1}(n) < B_{3,1}(n)$.*

Solution. The solution of this problem is put after 4.4.

4.4. ***a)*** **Statement.** *Prove that $D_{4,1}(3^k) = k + \log_2 k + c_{dk}$, where the sequence c_{dk} is bounded.*

Solution. We will prove that $\mathrm{D}_{4,1}(2^k) = k+t$ for the least t such that $2^t \geqslant k+t+1$ (then $t = \log_2(k+t)+O(1)$, hence $t = \log_2 k + O(1)$ as required). From the solution of 3.9 we already know that $\mathrm{D}_{4,1}(2^k) \geqslant k+t$. It remains to present an algorithm which enables to find the false coin in $k+t$ weighings.

Hereinafter, in all problems on testers the *significance* of a suspected coin means $i+1$ where i is the number of steps until the end of the process (by default) or until a specified moment of the process. The significance of any refuser is 1, and that of a standard is 0. The significance of a situation is the sum of significances of coins. Note that the sum of significances of possible situations after testing is not less than the initial significance; hence, for some result of the detection is decreases not more than twice.

Note that the significance of the initial situation is $(k+t+1)2^k \leqslant 2^{k+t}$. Hence to obtain $k+t$ steps we have to obtain diminishing twice (or near that) at each step. Thus we will construct an algorithm such that at any moment there exist refusers of not more than three distinct types, and the significance of the situation at m steps before the end of the process does not exceed 2^m. Obviously this hold at the initial moment.

First we describe a step of the algorithm roughly. Suppose there are m steps until the end of the process. We split the coins into groups so that (i) possible significances of situations after testing do not exceed 2^{m-1}, and (ii) each group includes refusers of not more than 2 types. After that we apply such tester that there are no refusers of its type. Suppose it indicates some group as including the false coin. Then the refusers after this step are just the former refusers from this group (of not more than two types) and the refusers of the tester in question (all of a single type), hence we have retained the required number of types of refusers.

Now we go in some detail. First we observe that, for the above distribution, the sum of significances of possible situations after weighing equals the initial significance. Hence if we are able to distribute the suspected coins so that these significances do not exceed 2^{m-1} then **at any rate** we can distribute the refusers so that this retains.

However we have to retain condition (ii) as well. This is provided by

Lemma on two buses. Given several students of three classes (let their quantities in the classes be $a_1 \leqslant a_2 \leqslant a_3$) and two buses (of capacities $b_1 \leqslant b_2$ where $a_1 + a_2 + a_3 \leqslant b_1 + b_2$). We have to distribute the students between the buses so that each bus contains students of not more than two classes. Then this is possible iff $b_1 \geqslant a_1$.

Proof. Suppose $b_1 \geqslant a_1$; we will present the required distribution. If the total capacity of the buses exceeds the number of students then we may decrease some of the numbers b_1 or b_2 by 1 so that the condition retains (if $b_2 > b_1$ then we decrease b_2, otherwise we decrease b_1). Thus we may assume $a_1 + a_2 + a_3 = b_1 + b_2$.

Put all the students in a line: first, all the students from the first class, then from the third, and after that from the second one. Then place first b_1 of them in the first bus, the next b_2 ones in the second bus. Then all students from the first class will be in the first bus (by the condition), and the students from the second class will be in the second bus (otherwise $b_2 < a_2$ and $b_1 > a_1 + a_3 > a_3 \geqslant b_2$, which is wrong).

Conversely, if $b_1 < a_1$ then $b_1 < a_2$ and $b_1 < a_3$. Thus no class can be put entirely into the first bus, hence the second bus will contain students from all three classes. □

Thus, if we know the number of refusers we have to place in each group then the lemma enables to determine the validity of condition (ii) in the distribution of refusers.

Now we proceed the construction of the algorithm. At the first k detections, we bisect both suspected coins and refusers; then obviously any result the significance decreases just twice, hence condition (i) holds. The condition of the lemma holds as well (since there is a type such that not more than a half of all refusers belongs to this type!), so we have managed in performing the step. After kth step of this kind we have only one suspected coins and some refusers of three types.

Now we consider subsequent steps in more detail. If at some moment there are no suspected coins then, on the same reasons, all the refusers can be split in two equal groups (or almost equal if their number is odd) so that condition (ii) holds. And since the significance did not exceed 2^m before this step, the number of coins in the groups does not exceed 2^{m-1}, so condition (i) holds as well.

It remains to consider the case when some suspected coin remains. Place it in the first group; then we may add to this group not more than $2^{m-1} - m$ refusers, and the rest of them will be in the other group. If the bus lemma does not apply then the number of refusers of each type exceeds $2^{m-1} - m + 1$. But the total significance of coins is now $(m+1)+3(2^{m-1}-m+1) = 2^m + (2^{m-1}-2m+4)$; it is not difficult to see that the expression in the brackets is always positive, so this situation is impossible.

b) **Statement.** *Prove that $B_{4,1}(3^k) = k + \log_3 k + c_{bk}$, where the sequence c_{bk} is bounded.*

Solution. Similarly we will prove that $\mathrm{B}_{4,1}(3^k) = k+t$ for the least t such that $3^t \geqslant 2(k+t)+1$ (then $t = \log_3 k + O(1)$). The estimate $\mathrm{B}_{4,1}(3^k) \geqslant k+t$ follows again from 3.9. We will construct the algorithm using the same argument (here the significance of a suspected coin in m steps before the end of the process equals $2m+1$). In this case, the conditions on distribution into three groups are as follows: (i) possible significances of situations after weighing do not exceed 3^{m-1}; (ii) each group includes refusers of not more than two types; (iii) some two groups (possibly after adding standards) contain equal number of coins (these will be two groups put on the scales).

Condition (ii) is guaranteed by the similar

Lemma on three buses. Given several students from three classes (let their numbers be $a_1 \leqslant a_2 \leqslant a_3$) and three buses (of capacities $b_1 \leqslant b_2 \leqslant b_3$ where $a_1+a_2+a_3 \leqslant b_1+b_2+b_3$). We have to distribute the students between the buses so that each bus contains students of not more than two classes. Then this is possible iff $b_1+b_2 \geqslant a_1$.

Proof. First we will present the distribution for the case $b_1+b_2 \geqslant a_1$. Again if $a_1+a_2+a_3 < b_1+b_2+b_3$ then we may decrease some of b_i so that the above condition holds.

Put all the students in a line: first all the students of the first class, then from the third, and after that from the second one. Then place first b_1 of them into the first bus, the next b_2 ones into the second bus, and all the remaining ones into the third bus. The first bus contains not more than one third of all the students, hence there are no students of the second class in it. The third bus does not contain students of the first class, by condition. And if the second bus contains students from all three classes, then it contains not less than a_3+2 students, and the third bus contains not more than $a_2-1 < a_3$ students. This is impossible since $b_2 \leqslant b_3$.

Conversely, if $b_1+b_2 < a_1$ then also $b_1+b_2 < a_2$ and $b_1+b_2 < a_3$. Thus no class can be put entirely into the first two buses. Hence the third bus contains students from all three classes. □

Remark. In fact, lemma on two buses is a special case of the above lemma when one bus contains no places.

Now we proceed with the algorithm. At the first k weighings we split both suspected coins and refusers into three equal parts; then all three conditions are satisfied. After the kth step of this kind we have a single suspected coin and some refusers of three types. Observe that after two first we already have $4 \cdot 3^{k-2}$ standards. This easily follows that condition (iii) is valid.

At subsequent steps, if to some moment no suspected coins remain, we can split all refusers into three almost equal groups so that condition (ii) is fulfilled; furthermore two groups include equal numbers of coins. Finally, suppose a suspected coin did remain. Then we put it into the first group; it is possible since its significance after this weighing is equal to $2m-1 \leqslant 3^{m-1}$. We may assume that we have to put equal number of refusers into the second and the third groups. Then the sum of two minimal capacities of groups is greater than one third of the total capacity, so the lemma applies. Condition (iii) holds as well. So the last case is done.

c) Try to find a better upper bound for these sequences.

4.3. Statement. *Find some value of n such that $B_{4,1}(n) < B_{3,1}(n)$.*

Solution. For example, $n = 3^{10}$ fits. The solution of 4.2b implies $\mathrm{B}_{4,1}(n) = 13$. In the sequel, we use terminology introduced in 3.9b).

Now we will prove that $\mathrm{B}_{3,1}(n) > 13$. Suppose the contrary; this is possible only if the significance decreases exactly thrice at each step (for any result of weighing!). This in turn means that at any moment there exist refusers of not more than two types, namely **not** of the type involved in the weighing in question.

Let us examine how this occurs. Denote a situation by a triple of numbers (a,b,c) where a is the number of suspected coins, and b and c are the numbers of refusers of two existing types. Then the initial situation is $(3^{10},0,0)$, and after the first weighing we have $(3^9, 2\cdot 3^9, 0)$. In the second weighing, let the numbers of suspected coins in the groups be a, b, c.

Suppose the weighing indicates the first group. Then we have the situation $(a, 3^9-a, t)$ where the number t is determined by the total significance: $23a + (3^9-a) + t = 3^{11}$ and so $t = 8\cdot 3^9 - 22a$. Note that $t \geqslant 0$, hence $a \leqslant \frac{4}{11}3^9$. Similarly $b, c \leqslant \frac{4}{11}3^9$, hence $a = 3^9 - (b+c) \geqslant \frac{3}{11}3^9$.

Consider the third weighing. If all a suspected coins are in the same group then its significance is not less than $21 \cdot \frac{3}{11}3^9 > 3^{10}$ which is impossible. Hence the suspected coins belong to at least two groups. To have refusers of not more than two types for each possible result, it is necessary to have refusers of a single type in each group. Thus one of the groups will contains **all** refusers of some type; let their number be s. If this group contains x suspected coins besides that then its significance after weighing equals $3^{10} = 21x + (a-x) + s = s + a + 20x$; thus $3^{10}-s-a$ is divisible by 20. Furthermore, we have either $s = 3^9 - a$ or $s = t = 8\cdot 3^9 - 22a$. In the first case $3^{10}-s-a = 2\cdot 3^9$ is not divisible by 20 which is impossible. Hence we have the second case, and then $3^{10}-s-a = 21a - 5\cdot 3^9$. Since 20 divides this number, 5 divides a. Similarly, 5 divides b and c; this contradicts $a+b+c = 3^9$ since the last number is not divisible by 5.

4.5. *a)* Statement. *Prove that $D_{x,1}(n) = D_{4,1}(n)$ for every n and $x > 4$.*

Solution. Suppose we must find the false coin among n coins. Note that if t satisfies the injequality $2^t < n(t+1)$ then we cannot find the false coin in t testing using any number of detectors on the reasons of significance. Similarly, if $2^{t-1} < \lceil n/2\rceil \cdot t + (n - \lceil n/2\rceil)$, then we also cannot find FC in t detections. Indeed, for any division for the first testing, some group contains not less than $\lceil n/2\rceil$ coins; and if the first testing indicates FC in this group then we cannot determine it on significance reasons. Clearly the second inequality is stronger than the first one, so it suffices to check only it.

Thus if we present an algorithm such that for any t with $2^{t-1} < \lceil n/2\rceil \cdot t + (n - \lceil n/2\rceil)$ it allows to find FC in t steps by 4 detectors then we have proved that $\mathrm{D}_{4,1}(n) = \mathrm{D}_{\infty,1}(n)$.

We will construct such algorithm which satisfies two conditions: (i) in i steps before the end of the process, the significance of the situation does not exceed 2^i; (ii) at each step, present are the refusers of not more than 3 types. At the first step we split coins into two almost equal groups (that is, groups of $\lceil n/2\rceil$ and $n - \lceil n/2\rceil$ coins). By assumption, the significance of the situation after weighing is $\leqslant 2^{t-1}$. At further steps we split suspected coins almost in half each time, and then we split refusers according to significance reasons, and the specific distribution of refusers is based on the lemma on buses.

Consider the ith step in more details. If the significance of the situation is less than 2^i then we add *dummy* refusers so that the new significance equals 2^i. It is not difficult to see that if the number of refusers (including dummy ones!) is not less than $3(i-2)-1$ then we clearly can provide that the significances of the situations possible after weighing differ not more than in 1. Then the significances in $i-1$ steps before the end of the process will not exceed 2^{i-1} in both cases because their sum does not exceed 2^i as required. Moreover in this case the numbers of refusers in two groups differ not more than twice, so the bus lemma applies here as well (check it yourself!).

It remains to show that the number of refusers in i steps before the end of the process is not less than $3(i-2)-1$, that is, the total significance of suspected coins does not exceed $2^i - 3(i-2) + 1$. It can be shown by some accurate calculation which we do not present here.

b) **Statement.** *Prove that $B_{x,1}(n) = B_{4,1}(n)$ for every n and $x > 4$.*

Solution. Similarly; in addition, we have to provide that two of groups obtained at any step contain equal number of elements. It is possible to do by adding standards.

Remark. The same methods (although with more technical details) enable us to show that the ideal number of testers does exist for any number of broken testers.

4.6. ***a)*** **Statement.** *Find whether the estimate of the same form as in problem 4.4 is valid for $D_{3,1}(n)$.*

Solution. Suppose we wish to find the false coin among 2^k coins using 3 testers one of which is broken.

Consider the minimal d such that $2^d + d + 1 \geqslant 2^k(2d+1)$. We will prove that d steps are sufficient to retain not more than two coins which can occur to be false (this will be the moment relative to which we determine the significance).

Put $x_i = 2^i + i + 1$. We construct a process such that for i weighings before the end the significance does not exceed x_i, and refusers always belong to not more than two distinct types. Note that if have provided this then the significance after the last step does not exceed 2, hence not more than 2 coins will be suspected or refusers at the end.

We construct the process by induction. Before the first weighing, both conditions are satisfied. Suppose we have to carry out i last weighings. We will define the first of them. Instead of the bus lemma, we use the following (obvious)

Lemma on dissection. Suppose we have some objects arranged in a line such that each of them costs not more than x, and the total price of them is S. Then the line can be cut into t parts so that each part costs not more than $\dfrac{S+(t-1)x}{t}$.

Without loss of generality, we may assume that there are refusers of the first and the second types. Put coins in a line: first the refusers of the first type, then the suspected coins, and then the refusers of the second type. Denote the number of suspected coins by a. Sum the refusers with coefficient 1, and the suspected coins with coefficient $i-1$. Then the sum of the coefficients does not exceed $x_i - 2a$; hence by lemma on dissection this line can be dissected into two parts such that the sum of coefficients in each part is $\leqslant \dfrac{x_i - 2a + i - 1}{2}$. If now we test coins of any part of the dissection, the significance of the part obtained equals the sum of its coefficients plus the number of suspected coins (since the suspected coins in this part are summed with coefficient i instead of $i-1$, and in the other part with coefficient 1 instead of 0). Thus for any result the significance does not exceed $\dfrac{x_i + i - 1}{2} = x_{i-1}$. Moreover if some part of the dissection includes refusers of both types then it includes all suspected coins; hence for each result of testing refusers of not more than two types can remain.

Thus after d weighings we have not more than two possible FC. It remains to observe that a single false coin among two coins can be determined in a bounded number of steps (for instance, in 3 steps). Hence we have proved that $d+3$ testings are sufficient. Furthermore $d = \log_2(2^k(2d+1) - d - 1)$ hence $d = \log_2 k + \log_2\log_2 k + O(1)$ as required.

b) **Statement.** *Тот же вопрос про $B_{3,1}(n)$.*

Solution. Similarly, we intend (for n sufficiently great) to keep only 3 possibly false coins among n coins in d weighings. We assert that it suffices to take d such that

$$(2d+1) + 2\cdot 3^d \geqslant n(2d+1).$$

Then (by problem) we will find FC in $d+3$ weighings; moreover $d = \log_3 n + \log_3\log_3 n + O(1)$.

Again, the moment for determining the significance is the moment after the dth weighing; then the initial significance equals $n(2d+1)$.

Denote $y_i = (2i+1) + 2 \cdot 3^i$; we arrange the process so that in i weighings before the end of the process the significance does not exceed y_i and the number of types of refusers does not exceed 2; then the significance after the dth weighing does not exceed 3 as required.

The step of the algorithm again is constructed similarly. Attach the factor $D = 2i - 2$ to all suspected coins, and the factor 1 to all refusers; then the total sum of factors does not exceed $y_i - 3a$ where a is the number of suspected coins. Put all coins in a line: first the refusers of the first type, then the suspected coins, and at last the refusers of the second type. Then the lemma on dissection implies that they can be split into three parts so that the sum of factors in each part does not exceed $\dfrac{y_i - 3a + 2D}{3}$. This means that the significance of any situation after weighing with these three parts does not exceed $\dfrac{y_i - 3a + 2D}{3} + a = y_{i-1}$ as required.

The last obstacle which might appear is that the numbers of coins in three groups obtained may differ. We will show that the number of standards is sufficient to equalize two groups: then we can arrange the required weighing. As is easily seen, this trouble does not exist at the first two steps: in the first weighing, two groups contain equal number of coins, and in the second weighing we can easily redistribute refusers and suspected coins in three groups so that two groups will contain equal number of coins (by a slight modification of the algorithm for dissection but with no weakening of the estimate). After these weighings, we have not less than $\frac{4}{9}n - 2$ standards. We have distributed not more than $\frac{5}{9}n + 2$ coins in 3 groups, hence the difference of numbers of coins in two of these groups does not exceed $\frac{5}{27}n + 1 \leqslant \frac{4}{9}n - 2$ for $n \geqslant 12$. Thus the number of standards is sufficient. The solution is complete.