Stability of intersections of paths in the plane

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He wants it rare, but something's there
that holds him back from the attack
Accept and Deaffy, Dogs on Leads.

The main results are criteria for stability of intersections of paths and cycles in the plane (problems D-2.d, D-3.d; the proof is outlined in the problems D-2.abc and D-3.abc; all the definitions are given below). The criteria are formulated in terms of derivation of graphs and cycles.[1]

In the following problems, let us agree that for any assertion formulated as a problem, a proof of this assertion is required.

The parts A, B, C are set before a half-way finish, the rest are set after it.

0. The city N in the plane is formed by several squares (of side 1) joined by disjoint roads (straight line segments). Suppose that there is a path passing through each road exactly once (this path can go through certain squares several times). Prove that there is a path *without self-intersections* passing through each road exactly once.

A. Stability of intersections of a pair of paths.[2]

Problem on the stability of intersections of a pair of paths. Two hunters hunt in a forest. Each leads a dog on a short lead. The dogs obey the hunters and move as they are commanded. If one dog intersects the path of another, it barks and scares off a game. Having fixed paths of the hunters, how to determine whether these hunters could maintain successful hunting (in this case the intersection of paths of the hunters is called *unstable*[3])?

We shall assume that all the paths of hunters and dogs are formed by finite unions of rectilinear arcs (i.e. are *piecewise-linear*). For such paths of hunters there is a slow full search algorithm for recognition of the stability of intersections. Finding a fast algorithm is an unsolved problem.

In the following problems we assume that the hunters move along a certain system of roads in a plane. We assume that both the hunter and the dog are points (not necessarily distinct). The lengths of leads are 1 m.

A-1. Two hunters move along the road that has the form of a segment of 1 km length. The hunters may change the direction of movement. Prove that independently of hunters' movement one dog could move without intersections with the path of another dog.

Example. Two hunters moved (with constant speed and permanent direction) along rectilinear roads of length 2 km; the roads intersect at their middle points and form right angles at the intersection point (fig. 1.a, where $\varphi(I_1)$ and $\varphi(I_2)$ are paths of hunters, and f_1 and f_2 are possible paths of dogs). Then one dog did intersect the path of the other.

A-2. A system of roads has the form of letter "H" (see fig. 1.b); the lengths of AB, BC, BE, DE, EF are 1 km. The first hunter moved along the path $ABEF$ and the second along $CBED$. Then one dog did intersect the path of the other.

[1]Problem of the stability of intersections of paths interesting not only with the point of view of graph theory but also with the point of view of topology. It is a particular case of problem on realization of mapping graphs in the plane. This sequence of problems is based on the articles [Mi97, Sk03] and overlaps with [RS00, §2, S, chapter 7] only in problems 0, B1 and D4.

[2]It is possible to solve many problems of this sequence by experiment. To carry out the experiments that involve dogs ask adults for permission.

[3]The formal definition is given in page 5.

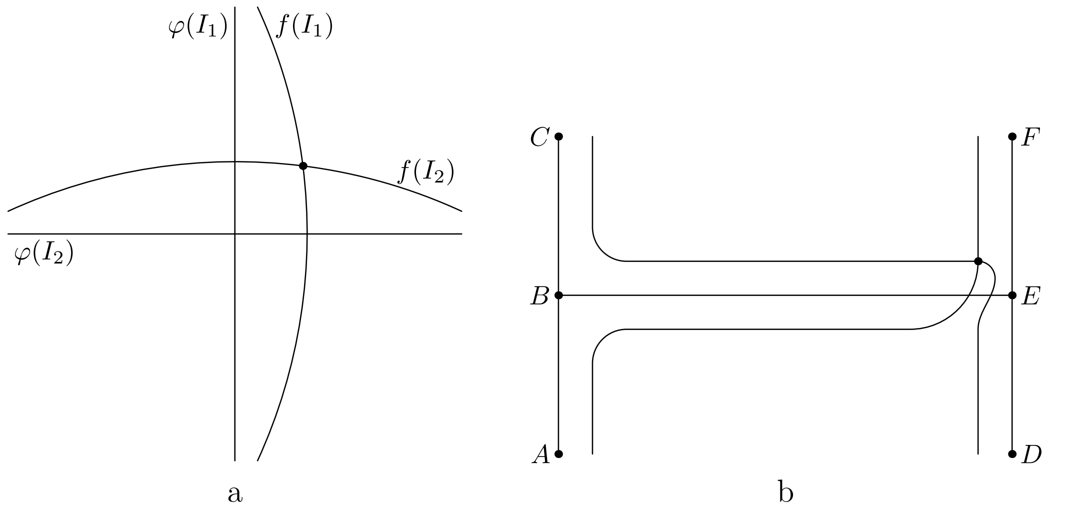

Figure 1: The transversal intersection and a path in letter "H"

For proving that certain paths of dogs must intersect the following theorem could be useful. You are allowed to use it without proof.

A *cycle* is a path whose starting point and endpoint are the same; it is not specified which point is the starting point (see Figure).

Note that *self-intersection* points (i.e. points corresponding to the dog's intersection of its own path) are not counted as *intersection* points.

Two piecewise-linear paths (or cycles) intersect *transversally*, if near each intersection point these paths look like two paths $\varphi(I_1)$ and $\varphi(I_2)$ in Fig. 1.

Even Number Theorem. *Two (piecewise linear) cycles in the plane that intersect transversally have an even number of intersection points.*

A-3. A system of roads has the form of letter "Y". This letter is formed by three straight-line segments making angles $2\pi/3$ at the common point (fig. 4.Y). Give an example of two hunters' paths such that the hunters could not maintain successful hunting.

B. Stability of self-intersections of path and of cycle.

Problem of the stability of self-intersections of path. One hunter walks in a forest and leads a dog on a short lead. The dog obeys the hunter and moves as it is commanded. If the dog intersects its path, it barks and scares off a game. How to determine, for the fixed path of the hunter, whether he could maintain successful hunting? (If he can, the self-intersection of the path of hunter is called **unstable**.)[4]

The main result of the presented problems is a fast algorithm recognizing the stability of self-intersections [5].

B-1. (a) A hunter moves along the road that has the form of a segment of length 1km. The hunter may change the direction of movement. Prove that independently of the hunter's movement the dog can move without intersecting its trace.

(b) The same for the road that has the form of the circle of radius 1km.

[4]The formal definition is given in page 5.

[5]Problem on the stability of self-intersections of paths is similar to the classical problem on planarity of graphs (i.e. realization of graphs in the plane without self-intersections) and even is reduced to recognition of the planarity of graphs (however the number of graphs required for one path is large). Problem of the realization of graphs is solved, for example, by the Kuratowski criterion. For the problem on approximation by embeddings such a criterion does not exist [Sk03], see problem D8.

B-2. (a) Each path without self-intersections has unstable self-intersections (this should not confuse the reader).

(b) If self-intersections of a path are unstable, then the same is true for all subpaths of this path [6].

(c) If intersections of a certain pair of subpaths of a path are stable, then self-intersections of this path are stable.

(d) There exists a path that does not contain transversal intersections but does have stable self-intersections.

One of the main results of this sequence of problems is the following theorem (try to prove it, but not too hard!)

Theorem on subpaths. *A path in the plane has stable self-intersections if and only if there are two subpaths of this path that have stable intersections.*

B-3. Assume that a system of roads form certain graph in the plane, so that the edges are straight line segments of length at least 1 km and the distance from every vertex to every edge that does not contain this vertex is more than 10 m. Suppose that the hunter walked along this system of roads, passing along every road only once and changing the direction of his movement only at vertices. Prove that the path of the hunter has stable self-intersections if and only if the path contains transversal self-intersections.

B-4. There exists an algorithm recognizing stability of self-intersections for given path in the plane.

B-5. (a) The hunter moved along a circular path of diameter 1 km (with constant speed and permanent direction along the circle) and winded twice along the circle. He lead a dog on a lead of length 1 m. The dog returned to the starting point at the end of the movement. Prove that the dog necessarily intersect its own path (in a certain moment of time different from the final movent, fig. 2).

(b) Is the analogue (a) correct if we do not suppose that the dog returned to the starting point at the final moment?

(c) Prove the analogue of (a) for the case when the hunter winded *three* times along the circle.

(d) For which number of windings in the analogue of (a) the dog necessarily intersected its own path?

(e) Suppose that the road is the segment of length 1 km. Prove that independently of the hunter's movement the dog can move without self-intersections and so that at the final moment of movement it will return to the starting point.

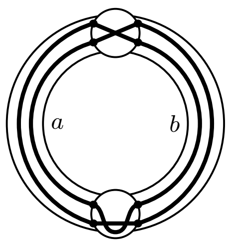

Figure 2: A path of the dog

C. Derivative of graphs and paths.

[6]The formal definition of a *subpath* is given in page 5

The *derivative* G' of a graph G is a graph whose vertices are in one-to-one correspondence with the edges of the graph G. Vertices e' and f' that correspond to edges e and f are connected by an edge in the graph G', if the edges e and f have a common vertex (fig. 3).

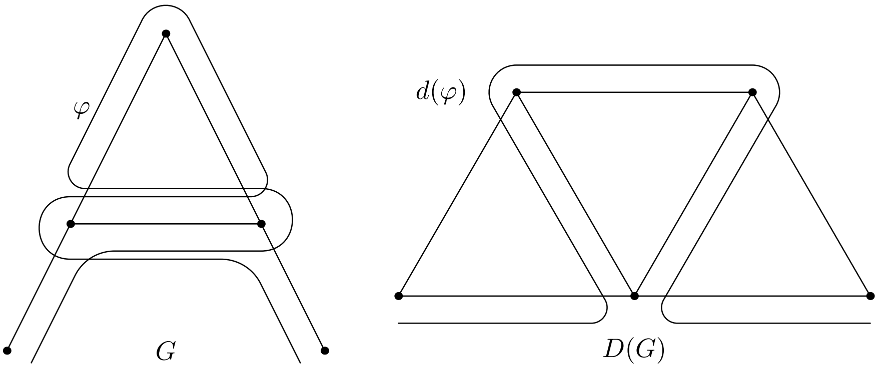

Figure 3: The derivative of a path in a graph

C-1. Draw the derivatives graphs of (fig. 4):
(a) an arc with n edges; (b) a circle with n edges;
(c) a star with n rays; (d) letter "H".

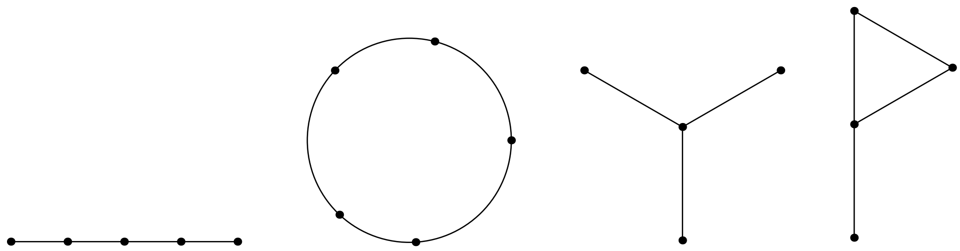

Figure 4: Differentiate us!

C-2. A graph is called *planar* if it can be drawn in the plane without self-intersections. The derivative of a planar graph is not necessarily planar.

A *path* in a graph G is a sequence of vertices $v_0, v_1, \dots, v_n$ such that v_i and v_{i+1} are connected by an edge in the graph G.

A path (a cycle) $v_0, \dots, v_n$ is called **an Euler path (cycle)** if it passes through each edge of the graph G only once, i.e. if each edge of the graph G is present among the edges $v_0v_1, v_1v_2, \dots, v_{n-1}v_n$ only once.

Suppose that a path φ in a graph G is defined by a sequence $v_0, v_1, \dots, v_n$ of vertices. Consider the sequence $(v_0v_1)', \dots, (v_{n-1}v_n)'$ of vertices of the derivative graph. In this sequence there could be the same consecutive vertices. Substitute each set of the same consecutive vertices by one vertex. In this way we obtain a path φ' that is called the *derivative* of the path φ. The derivative of a pair of paths is a pair of paths defined analogously.

Example. Let A be the 'letter A' graph, i.e. the graph with vertices a_1, a_2, a_3, a_4, a_5 and edges a_1a_2, a_2a_3, a_2a_4, a_3a_4, a_4a_5 (fig. 3). Denote by $b_1 = (a_1a_2)'$, $b_2 = (a_2a_3)'$, $b_3 = (a_2a_4)'$, $b_4 = (a_3a_4)'$, $b_5 = (a_4a_5)'$ the vertices of the derivative of A. Let φ be a path $a_1a_2a_4a_2a_3a_4a_2a_4a_5$ in the graph A. Then the derivative φ' is the path $b_1b_3b_2b_4b_3b_5$.

C-3. (a) Find the first and the second derivatives of the paths and the pairs of paths from fig. 3 and fig. 1 (and also of your paths constructed in problems A3 and B2d).

(b) Given a path with n vertices, the number of vertices in the derivative path does not exceed $n-1$.

(c) We say that a path φ has a *return point*, if $v_i v_{i+1} = v_{i+1} v_{i+2}$ for some integer i. A path (cycle) φ is an Euler path if and only if it does not have return points and φ' does not have self-intersections.

Formal definitions.

Let us give an equivalent formulation of problem B-5 (the equivalence is proved in [Mi97]). Consider two clearings (i.e. two disks) in the plane connected by two paths (i.e. strips) a and b, as it is shown on Fig. 2. The dog moved on the clearings and paths, and at last returned into the starting point. Each time when the dog moved from the clearing to the path, it wrote down the letter corresponding to this path. It is stated in the problem B-5 that if we get the word *abab* then the dog necessarily intersected its own path (at a certain moment of time different from the final moment).

The other problems can be reformulated in a similar way.

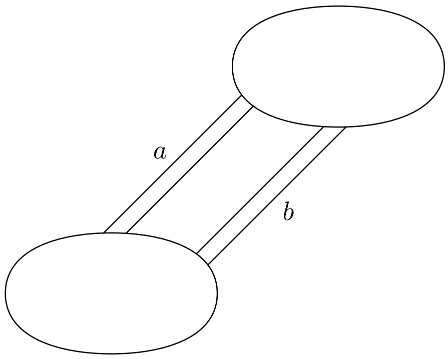

Figure 5: Two clearings

We shall give the formal definition of stability of intersections and self-intersections. Denote a segment by $I = [0,1]$, a circle (i.e. a segment with the ends glued) by S^1 and the plane by $\mathbb{R}^2$. *A piecewise-linear path* in the plane is a mapping $\varphi : I \to \mathbb{R}^2$ for which there exist points $0 = v_0 < v_1 < \ldots < v_n = 1$ such that φ is linear on each of the segments $[v_i, v_{i+1}]$. A *cycle* is defined analogously substituting I by S^1. We shall consider only piecewise-linear paths and we shall call them simply *paths*. A *subpath* of a path $\varphi : I \to \mathbb{R}^2$ is a path $\psi : J \to \mathbb{R}^2$ such that $\psi = \varphi|_J$, where $J \subset I$.

A path $\varphi : I \to \mathbb{R}^2$ has *unstable* self-intersections (or *allows the removal of self-intersections by a small perturbation*, or *is approximable by embeddings*), if there exists a path without self-intersections, arbitrarily close to our path (i.e. if for each $\varepsilon > 0$ there exists a path $\varphi : I \to \mathbb{R}^2$ without self-intersections such that the distance between the points $f(x)$ and $\varphi(x)$ is less than ε for each point $x \in I$). The stability of self-intersections of the *cycle* $\varphi : S^1 \to \mathbb{R}^2$ is defined analogously.

A pair of paths $\varphi_1, \varphi_2 : I \to \mathbb{R}^2$ has *unstable* intersections (or *allows the removal of self-intersections by a small perturbation*), if there exist a non-intersecting pair of paths arbitrarily close to our paths (i.e. if for each $\varepsilon > 0$ there exist non-intersecting paths $f_1, f_2 : I \to \mathbb{R}^2$ such that distance between the points $f_i(x)$ and $\varphi_i(x)$ is less than ε for each point $x \in I$ and for each $i = 1, 2$.

For instance, the *transversal intersection* (Fig. 1) of two paths is stable.

For example, the problem B-1ab can be reformulated in this language as follows:

If the image $\varphi(I)$ of a path $\varphi : I \to \mathbb{R}^2$ is a segment or a circle then the self-intersections are unstable.

Hints and solutions of some problems from parts A, B, C.

A-1. Draw a straight line along our road. It splits the plane into two half-planes. The first hunter orders his dog to move in one of the given half-planes, and the second hunter orders his dog to move in another half-plane. Then the dogs' paths do not intersect.

A-2. Assume that the dogs may move so as not to intersect each other's paths. Let $A'F'$ and $C'E'$ be the paths of the dogs. We close these paths by adding to them broken lines $F'XA'$ and $E'XC'$ shown in the figure 6. The maximal distance between the hunter and the dog is much less than the distances between each two of the points A, C, E, F, hence the broken line $F'XA'$ does not intersect the path $C'E'$, and the broken line $E'XC'$ does not intersect the path $A'F'$. Thus the two cycles $A'F'XA'$ and $C'E'XC'$ transversally intersect in the unique point X. And according to the Even number Theorem the number of their intersection points has to be even. The obtained contradiction proves that the dogs' paths do intersect.

Examples to *the problems A3 and B2d* are presented in Fig. 6, where (for clarity) is shown not the initial path itself, but a general position path close to the initial path is drawn. However, see [Mi97, Sk03].

Hint: it is possible to reduce this problem to non-planarity of the Kuratowski graphs K_5 and $K_{3,3}$. The dotted line on the Fig. 6 will help to do this.

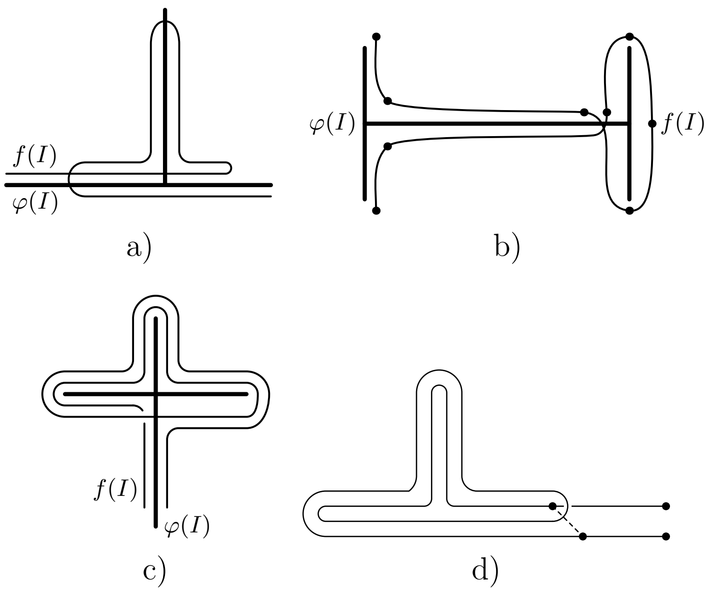

Figure 6: Paths with stable self-intersections

B-1. (a) Direct the axis $0x$ along the road, and take an axis $0y$ perpendicular to the road. Assume that the hunter's coordinate (expressed in meters) is given by the function $x(t)$, and the hunting time changes in the range from 0 to T. The hunter orders the dog to move along the graph of the function $x(t)$, compressed towards the axis $0x$, i.e. let the dog's coordinates at the moment t be $(x(t), t/T)$. It is easy to see that in this case the dog will not intersect its own path, and in every moment of time it will be closer than 1 m from the hunter.

(b) We will act analogously to the item (a): the dog moves so that in each moment of time it is on the ray that is directed from the center of the circle to the point where the hunter is situated, and its distance from the hunter is t/T m (where T is the total hunting time). Then the dog will not intersect its own traces, and in every moment of time it will be closer than 1 m from the hunter.

B-2. (a) Suppose that the dog follows in the tracks of the hunter, i.e. in each moment of time the dog and the hunter are in the same point (it is not prohibited by the conditions). The hunter's path will not self-intersect, and thus the dog's path also will not self-intersect.

(b) The self-intersections of the path are unstable, and so the dog can move without intersecting its own path. We shall consider the dog's movement only in the segment of time that corresponds to the chosen subpath. It also does not self-intersect. That's why the chosen subpath has unstable self-intersections too.

(c) Suppose that self-intersections of the path are unstable. Then the dog may move without intersecting its path. Concern the dog's movement only in two segments of time that correspond to the chosen subpaths. These two paths do not intersect. Thus self-intersections of the corresponding pair of subpaths are unstable.

B-3. It is clear that if the hunter's path contains transversal self-intersection then the hunter's self-intersections are stable. Prove that if the path does not contain transversal self-intersections in the considered case then the hunter's self-intersections are unstable. We shall draw a circle with the radius 1 m around every point of connection of roads. Let the dog follows the tracks of the hunter all the time when the hunter is outside of these circles. At the moment when the hunter begins to move inside a certain circle, the dog takes a short cut, moving along the chord instead of a pair of radii, as it is shown on the figure. The built path does not self-intersect. Indeed, the hunter passes along each road only once, so the dog does not intersect its own path outside the circles. If the dog intersects its path inside a circle then some two of chords drawn by us intersect. And it is possible only if the hunter's path has a transversal self-intersection in this point.

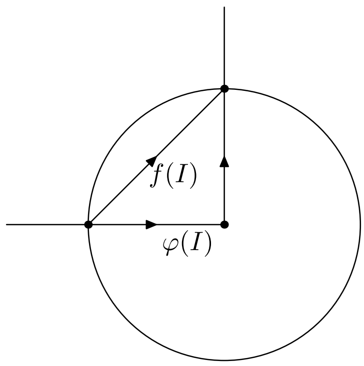

Figure 7: The dog takes a short cut

B-4. See the formulation of the problem D7 (simple but slow algorithm) and Minc theorem D2d (more complex but fast algorithm).

B-5. (a) Assume that the dog may move without intersecting its own traces. Let A be a certain point of the circle. Consider the ray OA that is directed from the center of the circle to the point A. It is obvious that the dog's cycle has intersected the ray OA at least twice: at least once when the hunter made the first cycle and at least once when he made the second cycle. We mark on this ray all its intersection points with the dog's path. It is obvious that there exist two "adjacent" marked points A' and A'' (i.e. such points that there are no marked points on the segment between these two ones), one of them is related to the moment when the hunter made his first turn along the circle, and the other - to the moment when the hunter made the second turn. Let us "tear" the dog's cycle in the points A' and A'' and add a pair of paths p' and p'' situated "near" the segment $A'A''$ which intersect transversally in the point X, as it is shown on the figure. As a result, we will obtain from the dog's cycle a pair of cycles that intersect transversally in the unique point X. But, according to the Even number Theorem, the quantity of the intersection points of two cycles must be even. The obtained contradiction proves that the dog must intersect its own path.

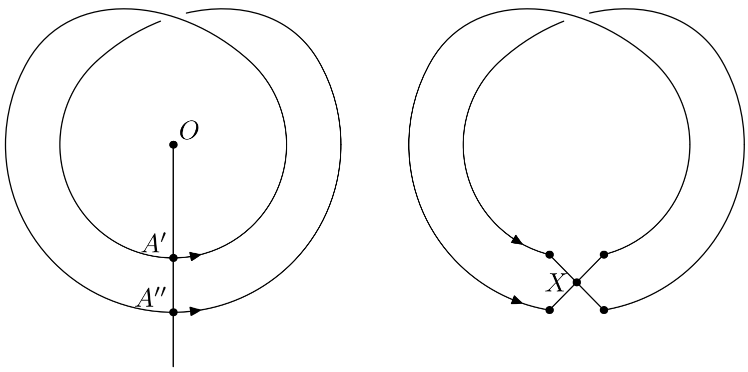

Figure 8: Transformation of the cycle

(b) Answer: in this case the dog may move without intersecting its own path.

(c) The proof repeats word-for-word our reasoning from the item (a).

(d) Answer: for every number of turns (in any direction) not less than two. The proof repeats word-for-word our reasoning from the item (a).

(e) We may consider (without lack of generality) that the hunter was in all of the points of the road, including its ends (otherwise we shall just diminish the segment of the road to obtain this condition). In this case it is possible to suppose that the hunter's cycle begins and finishes at one of the ends of the segment. Let the dog move along the graph of the hunter's movement, compressed towards the road (analogously to the solution of the problem B1a). Then we have only to close the dog's path, adding a broken line situated near the end of the segment, as it is shown on the figure. The constructed cycle does not self-intersect.

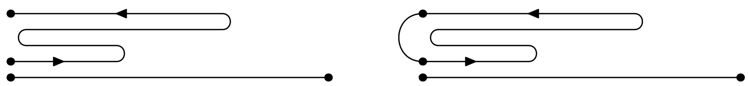

Figure 9: We close the dog's path

C-1. (a) Arch with $n-1$ edges; (b) circle with n edges; (c) full graph with n vertices; (d) graph made from the two triangles with a unique common vertex.

C-2. Example: a star with 5 rays - a planar graph which derivative is a non-planar graph (a full graph with 5 vertices).

C-3. (a) E. g., see fig. 10.

(b) Let the initial path consist of n vertices $v_1, v_2, \ldots, v_n$. Then there are exactly $n-1$ vertices in the sequence $(v_1v_2)', (v_2v_3)', \ldots, (v_{n-1}v_n)'$. To construct an arbitrary path we (maybe) exclude some vertices from the given path. As a result we get the path containing not more than $n-1$ vertices.

(c) Let ψ be the Euler path $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$. Since the Euler path comes along every edge only once, there are not two same edges in the sequence $(v_1v_2), (v_2v_3), \ldots, (v_{n-1}v_n)$. Thus in the sequence of the vertices of the derivative of the given path $(v_1v_2)', (v_2v_3)', \ldots, (v_{n-1}v_n)'$ there are not two same vertices. So the path ψ' does not self-intersect.

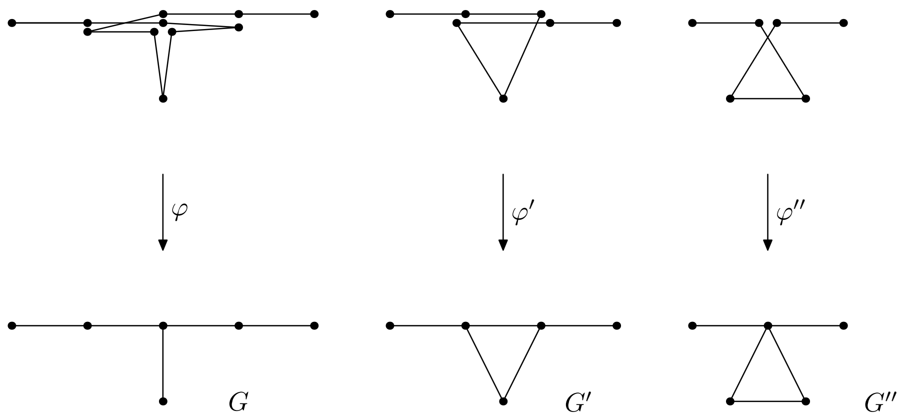

Figure 10: The second derivative of a path

D. Main problems

D-1. Self-intersections of the paths in Figure 6 are stable.

For a piecewise linear path $\varphi : I \to \mathbb{R}^2$ the image $\varphi(I)$ can be considered as a graph with vertices $\varphi(v_1), \dots, \varphi(v_n)$. From every piecewise linear path φ a path in the graph $\varphi(I)$ can be constructed in a unique way. Vice versa, suppose that the graph G be drawn in the plane without self-intersections so that all its edges are straight line segments. For every path $v_1, \dots, v_n$ in the graph G we shall build a path $\varphi : [0,1] \to G$ in the plane, setting $\varphi(\frac{i}{n}) = v_i$ for every $i = 0, \dots, n$ and extending φ linearly on the segments $[\frac{i}{n}, \frac{i+1}{n}]$. The path $\varphi : [0,1] \to \mathbb{R}^2$ that is constructed in this way from a certain path in the graph G, is called *simplicial.*

D-2. (a) If a path φ does not contain transversal self-intersections then the graph $\varphi'(I)$ is planar.

Fix the "natural" embedding of the graph $\varphi'(I)$ into the plane (invent the definition yourself). Then $\varphi' : I \to \varphi'(I)$ is a certain path in the plane.

(b) If self-intersections of a path φ are unstable then self-intersections of the path φ' are also unstable.

As a corollary, obtain the proof of the statement from problem A2.

(c) If a path φ does not contain transversal self-intersections and self-intersections of the path φ' are unstable then self-intersections of the path φ are also unstable.

(d) *The Minc Theorem.* Self-intersections of a simplicial path $\varphi : I \to G$ containing n points are stable if and only if for a certain $k = 0, \dots, n$ its k-th derivative $\varphi^{(k)}$ contains a transversal self-intersections.

(e) Prove the Theorem on two subpaths.

D-3. (a) For which m self-intersections of the cycle winding of degree m (Figure 2 for $m = 2$) are stable?

(b) For every cycle φ there is k such that $\varphi^{(k)}$ is a winding.

(c) The statements of the problems D2abc remain true for a cycle φ.

(d) *Theorem.* The self-intersections of a simplicial cycle $\varphi : S^1 \to G$ that contains n points are stable if and only if for certain $k = 0, \dots, n$ its k-th derivative $\varphi^{(k)}$ either contains a transversal self-intersection or is a standard winding of degree $m \neq 0, \pm 1$.

D-4.* Consider a cycle φ. How to determine m such that $\varphi^{(\infty)}$ is a winding of degree m?

D-5.* Consider a set of paths in a given graph in the plane. Formulate and prove a criterion of approximating this set by a set of

(a) non-intersecting and non-self-intersecting paths;

(b)* non-intersecting (but possibly self-intersecting) paths.

D-6. How to construct an "integral" of a given path? Use this to invent new examples of paths with stable self-intersections.

D-7. (a) For a simplicial path $\varphi : I \to G \subset \mathbb{R}^2$ substitute every edge of the graph G by k close edges if the path φ passes along this edge k times. Denote the constructed graph by $\tilde{G} \subset \mathbb{R}^2$. Denote by $\pi : \tilde{G} \to G$ the projection mapping to an edge ab the union of multiple edges corresponding to the edge ab of G. Self-intersections of the path φ are stable if and only if there exists a path $\psi : I \to \tilde{G}$ without transversal self-intersections such that $\pi \circ \psi = \varphi$.

(b)* Invent a fast algorithm for recognition whether a given path in the graph in the plane has transversal self-intersections.

D-8. There exists an infinite set of paths $\varphi : I \to \mathbb{R}^2$ with stable self-intersections such that their images:

(a) are trees (containing each other);

(b) are "letter Y" (see the Figure 4.Y);

(c) are (A. Chalyavin) "letter P" (see the Figure 4.P), the paths do not have return points and no one of them is a subpath of any other.

D-9.* The *minor* of a graph is a graph that is obtained from the initial graph by several operations of throwing away (the interior of) an edge or gluing ends of an edge. The Kuratovski Theorem has the following equivalent formulation: a graph is planar if and only if it does not have minors isomorphic to K_5 and $K_{3,3}$.

Invent a notion of a minor of a path and find out whether there exists an infinite set of (piecewise linear) paths $\varphi : [0, 1] \to \mathbb{R}^2$ with stable self-intersections so that no path of these is a minor of any other graph from this set. Solve the same problem for the set of paths whose images are triods or whose images are trees embedded into each other.

Hints and solutions of some main problems.

D-1. Follows by **D2b**.

D-2. (a) Draw the graph $\varphi'(I)$ in the plane as follows. Put the vertices of the graph $\varphi'(I)$ in the middle points of corresponding edges of the initial graph G. Edges of the graph $\varphi'(I)$ are drawn as follows.

For each vertex v of the graph G make the following construction. Draw a small circle in the plane centered at this point v. Every edge of the graph G issuing out of the vertex v intersects this circle at a certain point. Go clockwise along the circle. Enumerate the edges of the graph G issuing out of v in their order along the circle: $v_1, v_2, \dots, v_n$. Near the intersection point of the edge v_i and the circle take $n-1$ points $v_{i,1}, v_{i,2}, \dots, v_{i,i-1}, v_{i,i+1}, \dots, v_{i,n}$ in the circle in this order *counterclockwise*.

For each edge $v_i'v_j'$ of the graph $\varphi'(I)$ make the following construction. Draw the edge as a broken line of three segments, so that the edge connects the middle point of the edge v_i with the middle point of the edge v_j and passes through the points $v_{i,j}$ and $v_{j,i}$.

We shall show that the drawn graph $\varphi'(I)$ does not have self-intersections. In the opposite case the self-intersection point is inside one of the constructed circles. Thus for certain edges $v_i'v_j'$ and $v_k'v_l'$ in the graph $\varphi'(I)$ the segments $v_{i,j}v_{j,i}$ and $v_{k,l}v_{l,k}$ intersect. But this is possible only if the initial path has a transversal self-intersection.

The same solution can be obtained by construction of the system of discs and strips for the graph $\varphi'(I)$ (cf. page 5). Figure 11 may help to realize this idea.

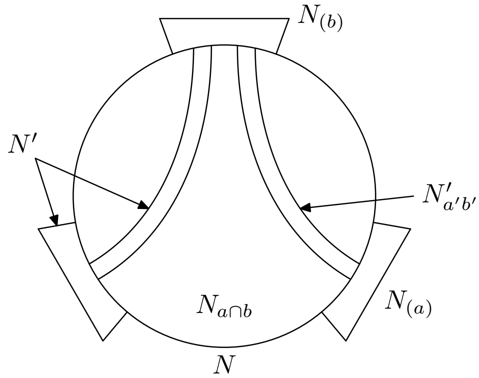

Figure 11: Discs and strips for the derivative graph

(b) A detailed solution can be found in the article [Sko03, Lemma 2.2A].

(c) The given statement is proved using an analogous method. A detailed proof is given in the article [Sko03, Lemma 2.1].

(d) The Minc theorem is deduced from the previous two items in the following way: first, if self-intersections of the path are unstable then by (b) all the derivatives of the path also have unstable self-intersections. Second, certain derivative of the path is a path which consists of a unique edge, because the quantity of edges strictly decreases when we differentiate the path. So if each derivative does not have transversal self-intersections then by (c) self-intersections of the initial path are also unstable.

(e) *Hint.* Suppose that self-intersections of the path are stable. Then there exists certain derivative having a transversal self-intersection. The transversal self-intersection is formed of 2 paths containing 2 edges each. Take two subpaths of the initial path whose derivatives are these paths. Then their self-intersection is stable.

D-3. (a) Answer: Each m except $\{-1, 0, 1\}$. A detailed solution is given in the article [Sko03, Lemma 2.3].

(c, d) Analogously to the items (c) and (d) of the previous problem. A detailed proof is given in the article [Sko03, Lemma 2.2.A and Lemma 2.1]

It is interesting to generalize these results to the case of maps $\varphi : K \to G \subset \mathbb{R}^2$, where the graph K is arbitrary. This case is treated in [Sko03].

D-5. (a) A theorem analogous to the Minc theorem D2d is true. The proof is analogous.

(b) The analogue of the Minc theorem is not true in this situation. A counterexample is shown in Fig. 12. Finding a fast algorithm for checking the stability of the intersection of two paths is an open problem.

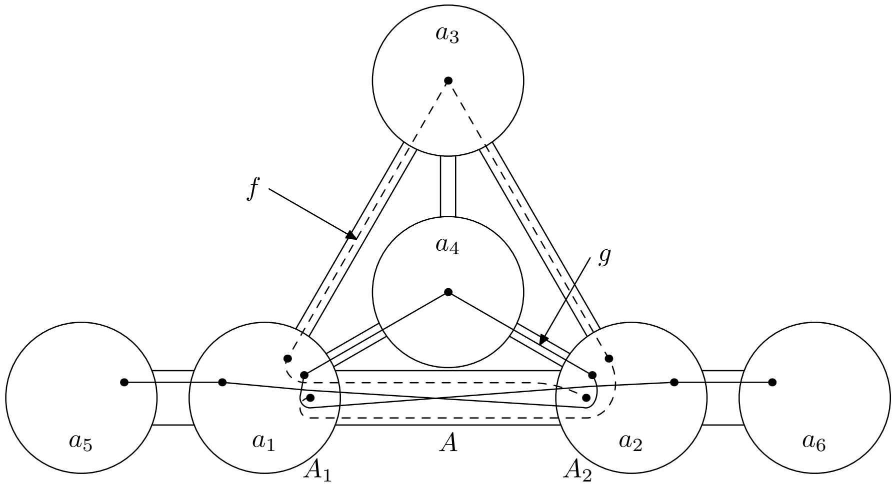

Figure 12: A pair of paths with stable intersection

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