ON APPROXIMABILITY BY EMBEDDINGS OF CYCLES IN THE PLANE

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Abstract. We obtain a criterion for approximability of piecewise linear maps $S^1 \to \mathbb{R}^2$ by embeddings, analogous to the one proved by Minc for piecewise linear maps $I \to \mathbb{R}^2$.

Theorem. *Let $\varphi : S^1 \to \mathbb{R}^2$ be a piecewise linear map, which is simplicial for some triangulation of S^1 with k vertices. The map φ is approximable by embeddings if and only if for each $i = 0, \dots, k$ the i-th derivative $\varphi^{(i)}$ (defined by Minc) neither contains transversal self-intersections nor is the standard winding of degree $\notin \{-1, 0, 1\}$.*

We deduce from the Minc result the completeness of the van Kampen obstruction to approximability by embeddings of piecewise linear maps $I \to \mathbb{R}^2$. We also generalize these criteria to simplicial maps $T \to S^1 \subset \mathbb{R}^2$, where T is a graph without vertices of degree > 3.

1. Introduction

A PL map $\varphi : K \to \mathbb{R}^2$ of a graph K is *approximable by embeddings* in the plane, if for each $\varepsilon > 0$ there is an ε-close to φ map $f : K \to \mathbb{R}^2$ without self-intersections. In the major part of this paper we consider the case when φ is either a path or a cycle, i. e. either $K \cong I$ or $K \cong S^1$.

Example 1.1. [12] The standard d-winding $S^1 \to S^1 \subset \mathbb{R}^2$ is approximable by embeddings in the plane if and only if $d \in \{-1, 0, 1\}$.

It can be also proved that a simplicial map $S^1 \to S^1$ is approximable by embeddings if and only if its degree $d \in \{-1, 0, 1\}$ (see Theorem 1.3). A *transversal self-intersection* of a PL map $\varphi : K \to \mathbb{R}^2$ is a pair of disjoint arcs $i, j \subset K$ such that φi and φj intersect transversally in the plane.

Example 1.2. An Euler path or cycle in a graph in the plane is approximable by embeddings if and only if it does not have transversal self-intersections (hence any Euler graph in the plane has an Euler cycle, approximable by embeddings).

The notion of approximability by embeddings appeared in studies of embeddability of compacta into $\mathbb{R}^2$ (see [12, 14, 11], for recent surveys see [7, §9], [2, §4], [8, §1], we return to this topic in the end of §1.) There exists an algorithm of checking whether a given simplicial map is approximable by embeddings (see [13], or else Simple-minded Criterion 4.1 below). A more convenient to apply criterion for approximability by embeddings of a simplicial path in the plane was proved in [6] (Theorem 1.3.I below, generalizing Example 1.2). The main result of this paper is an analogous criterion for approximability by embeddings of a cycle in the plane (Theorem 1.3.S below, also generalizing Example 1.2). These criteria assert that, in some sense, transversal self-intersections are the only obstructions to approximability by embeddings. Clearly, this is not true literally [12], and there is no Kuratowsky-type criterion.

We state our criterion (Theorem 1.3) in terms of *the derivative* of a path [5], [6, "*the operation d*"]. Let us give the definition (Fig. 1). First let us define *the derivative* G' of a graph G (it is a synonym for *line graph* and *dual graph*). The vertex set of the graph G' is in 1-1 correspondence with the edge set of G. For an edge $a \subset G$ denote by $a' \in G'$ the corresponding vertex. Vertices a' and b' of G' are joined by an edge if and only if the edges a and b are adjacent in G. Note that the derivatives G' and H' of homeomorphic but not isomorphic graphs G and H are not necessarily homeomorphic.

Now let φ be a path in the graph G given by the sequence of vertices $x_1, \dots, x_k \in G$, where x_i and x_{i+1} are joined by an edge. Then $(x_1x_2)', \dots, (x_{k-1}x_k)'$ is a sequence of vertices of G'. In this sequence replace each segment $(x_ix_{i+1})'$, $(x_{i+1}x_{i+2})$, $\dots$, $(x_{j-1}x_j)'$ such that $(x_ix_{i+1})' = (x_{i+1}x_{i+2})' = \dots = (x_{j-1}x_j)'$ by a single vertex. The obtained sequence of vertices determines a path in the graph G'. This path φ' is called *the derivative* of the path φ.

A 5-od (the cone over 5 points) is a planar graph whose derivative is the Kuratowsky graph, which is not planar. But if $G \subset \mathbb{R}^2$ and the path φ does not have transversal self-intersections, then the image of the map φ' is a planar subgraph $G'_\varphi \subset G'$ (we give the construction of a natural embedding $G'_\varphi \to \mathbb{R}^2$ in §2, Definition of N'). Change G' to the image G'_φ and φ' to its onto restriction $\varphi' : I \to G'_\varphi$. Define the k-th derivative $\varphi^{(k)}$ inductively. For a cycle φ the definition of *the derivative* cycle φ' is analogous.

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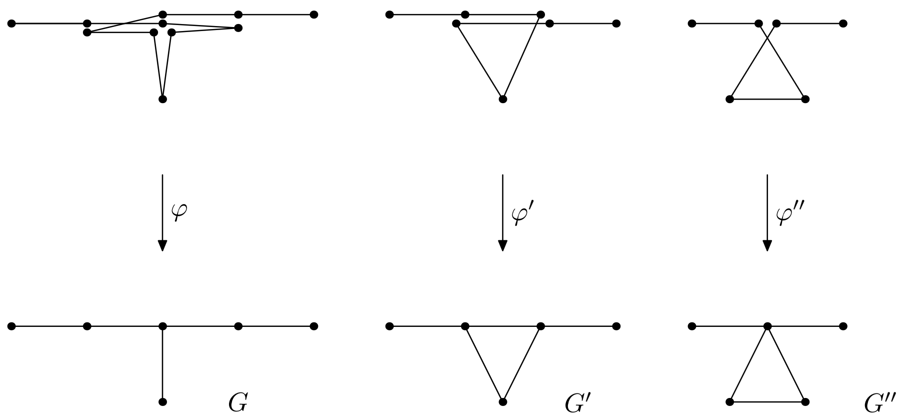

FIGURE 1. Derivatives of graphs and paths

An example to be used in the sequel is that $\varphi' = \varphi$ for a standard d-winding $\varphi : S^1 \to S^1$ with $d \neq 0$. Clearly, φ' is an embedding for any Euler path or cycle φ. Thus Example 1.2 is indeed a specific case of the following theorem.

Theorem 1.3. *I)* [6] *Let $\varphi : I \to \mathbb{R}^2$ be a PL map, which is simplicial for some triangulation of I with k vertices. The map φ is approximable by embeddings if and only if for each $i = 0, \dots, k$ the i-th derivative $\varphi^{(i)}$ does not contain transversal self-intersections.*

S) Let $\varphi : S^1 \to \mathbb{R}^2$ be a PL map, which is simplicial for some triangulation of S^1 with k vertices. The map φ is approximable by embeddings if and only if for each $i = 0, \dots, k$ the i-th derivative $\varphi^{(i)}$ neither contains transversal self-intersections nor is the standard winding of degree $d \notin \{-1, 0, 1\}$.

We prove both 1.3.I and 1.3.S in §2. Our proof of 1.3.I is simpler than the one given in [6].

In §3 we apply Theorem 1.3 to prove the following criterion.

Corollary 1.4. *A PL map $\varphi : I \to \mathbb{R}^2$ is approximable by embeddings if and only if one of the following equivalent conditions holds:*

D) (the deleted product property) There is a map $\{(x, y) \in I \times I\ x \neq y\} \to S^1$ such that its restriction to the set $\{(x, y) \in I \times I : \varphi x \neq \varphi y\}$ is homotopic to the map given by the formula $\tilde{\varphi}(x, y) = \frac{\varphi x - \varphi y}{\|\varphi x - \varphi y\|}$;

V) the van Kampen obstruction (defined in §3) $v(\varphi) = 0$.

The criterion 1.4.V, although more difficult to state, is more easy to apply than 1.3.I and 1.4.D. In Corollary 1.4 the arc I cannot be replaced by S^1: the standard 3-winding is a counterexample [8]. Obstructions like 1.4.D and 1.4.V appear in the related theory of approximability by link maps, i. e. by maps with disjoint images, but the criteria analogous to 1.3.I and 1.4.DV are not true (Example 3.3 below). In §3 we show that the n-dimensional generalizations of conditions 1.4.D and 1.4.V are equivalent for any PL maps $\varphi : K^n \to \mathbb{R}^{2n}$ (Proposition 3.2 below).

In §4 we generalize criteria 1.3 and 1.4 to PL maps $\varphi : K \to \mathbb{R}^2$, where K is an arbitrary graph [16]. We prove the following theorem (see Definition of the derivative in §2).

Theorem 1.5. *Let T be a graph without vertices of degree > 3. Suppose that T has k vertices. A simplicial map $\varphi : T \to S^1 \subset \mathbb{R}^2$ is approximable by embeddings if and only if the van Kampen obstruction $v(\varphi) = 0$ and $\varphi^{(k)}$ does not contain standard windings of degree $d \neq \pm 1$, d odd.*

Conjecture 1.6. Theorem 1.5 is true for a simplicial map $\varphi : K \to G \subset \mathbb{R}^2$, where K and G are arbitrary graphs.

If Conjecture 1.6 is true, then *a simplicial map $\varphi : T \to \mathbb{R}^2$ of a tree T is approximable by embeddings if and only if $v(\varphi) = 0$* [2, Problem 4.5].

Conjecture 1.7. A piecewise linear path $\varphi : I \to \mathbb{R}^2$ is approximable by embeddings if and only if for each pair of arcs $I_1, I_2 \subset I$ such that $I_1 \cap I_2 = \emptyset$ the pair of restrictions $\varphi : I_1 \to \mathbb{R}^2$ and $\varphi : I_2 \to \mathbb{R}^2$ is approximable by link maps (i. e. maps with disjoint images).

We conclude §1 by some words on the history of the notion of approximability by embeddings. We define the decomposition of a 1-dimensional compactum into *an inverse limit* and show how the notion of approximability by embeddings appears in studies of planarity of this compactum. We do not use this definition in our paper. To give an example, let us construct *the 2-adic van Danzig solenoid.* Take a solid torus $T_1 \subset \mathbb{R}^3$. Let $T_2 \subset T_1$ be a solid torus going twice along the axis of the torus T_1. Analogously, take $T_3 \subset T_2$ going twice along the axis of T_2. Continuing in the similar way, we obtain an infinite sequence of solid tori $T_1 \supset T_2 \supset T_3 \supset \dots$ The intersection of all tori T_i is a 1-dimensional compactum and is called *the 2-adic van Danzig solenoid.* By *the inverse limit* of an infinite sequence of graphs and simplicial maps between them $K_1 \xleftarrow{\varphi_1} K_2 \xleftarrow{\varphi_2} K_3 \xleftarrow{\varphi_3} \dots$ we mean the compactum

$$C = \{\, (x_1, x_2, \dots) \in l_2\ :\ x_i \in K_i \text{ and } \varphi_i x_{i+1} = x_i \,\}.$$

One can see from our construction that for the van Danzig solenoid all $K_i \cong S^1$ and all φ_i are 2-windings. It can be proved that any 1-dimensional compactum can be represented as an inverse limit. Such representation shows that any 1-dimensional compactum can be embedded into $\mathbb{R}^3$. It also gives an easy sufficient condition to planarity: for each positive integer i there should exist an embedding $f_i : K_i \to \mathbb{R}^2$ such that the map $f_i \circ \varphi_i$ is approximable by embeddings and f_{i+1} is 2^{-i}-close to $f_i \circ \varphi_i$.

2. Proofs

Theorem 1.3 follows from Example 1.1 and Lemmas 2.1, 2.2.A (for $K \cong I, S^1$) and 2.3, which are interesting results in themselves.

Lemma 2.1. *(for $K \cong I$ see* [6]*) Suppose that a simplicial map $\varphi : K \to G \subset \mathbb{R}^2$ of a graph $K \cong S^1$ or $K \cong I$ does not have transversal self-intersections. If φ' is approximable by embeddings, then φ is approximable by embeddings.*

Lemma 2.2. *A)* [6] *If a simplicial map $\varphi : K \to G \subset \mathbb{R}^2$ is approximable by embeddings, then the map φ' is approximable by embeddings.*

V) If a simplicial map $\varphi : K \to G \subset \mathbb{R}^2$ is approximable by mod 2*-embeddings, then the map φ' is approximable by* mod 2*-embeddings.*

Here *a* mod 2*-embedding* is a general position map $f : K \to \mathbb{R}^2$ such that for each pair a, b of disjoint edges of K the set $fa \cap fb$ consists of an even number of points. Definition of the derivative φ' needed for Lemma 2.2 is presented below

Lemma 2.3. *Let $\varphi : S^1 \to G$ be a PL map, which is simplicial for some triangulation of S^1 with k vertices. Then either the domain of $\varphi^{(k)}$ is empty or $\varphi^{(k)}$ is a standard winding of degree $d \neq 0$.*

This number d can be considered as the generalization of *the degree* of any simplicial map $S^1 \to G$. So it is interesting to get the solution of the following problem (it may also make criteria 1.3 and 1.5 more easy to apply).

Problem 2.4. Find an easy algorithm for calculation of the degree of the winding $\varphi^{(\infty)}$ for a given PL map $\varphi : S^1 \to G$.

Futher we use the following generalization of the definition of the derivative of a path stated in §1.

Definition (Definition of the derivative [6], see Fig. 1 and a part of Fig. 4)**.** First let us construct the graph K'_φ, which is the domain of the derivative φ'. By *a φ-component* of the graph K we mean any connected component α of $\varphi^{-1}a$ mapped *onto* a, for some edge $a \subset G$. The vertex set of K'_φ is in 1-1 correspondence with the set of all φ-components. For a φ-component $\alpha \subset K$ denote by $\alpha' \in K'_\varphi$ the corresponding vertex. Vertices α' and β' are joined by an edge in K'_φ if and only if $\alpha \cap \beta \neq \emptyset$. The *derivative* $\varphi' : K'_\varphi \to G'$ is a simplicial map defined on the vertices K'_φ by the formula $\varphi'\alpha' = (\varphi\alpha)'$. Change φ' to its onto restriction $\varphi' : K'_\varphi \to \varphi' K'_\varphi$. (In the original definition [6] G' is denoted by $D(G)$, φ' by $d[\varphi]$ and K'_φ by $D(\varphi, K)$.)

Proof of 2.3. We say that a simplicial map $\varphi : K \to G$ is *ultra-nondegenerate*, if for each edge $a \subset K$ the image φa is an edge of G and for each pair $a, b \subset K$ of adjacent edges we have $\varphi a \neq \varphi b$. Denote by $|K|$ the number of vertices in a graph K. Clearly, if $K \cong S^1$, then $|K'_\varphi| \leq |K|$, and $|K'_\varphi| = |K|$ only if φ is ultra-nondegenerate. Therefore it suffices to prove the lemma for this latter case (because the cases $K'_\varphi \cong I$ or K'_φ is a point are trivial). In this case the lemma is obvious, but we give the proof.

Let us prove that if an ultra-light simplicial onto map $\varphi : K \to G$ of the graph $K \cong S^1$ is not a standard winding of a nonzero degree, then $|G'| > |G|$. Note that for ultra-nondegenerate $\varphi : S^1 \to G$ the graph G does not contain hanging vertices. If the degree of each vertex of G is two, then φ is an ultra-nondegenerate simplicial map $S^1 \to S^1$, consequently φ is a standard winding, that contradicts to our assumption. So G contains a vertex of degree at least 3. Then by the above the number of edges of G is greater than the number of vertices, hence $|G'| > |G|$. Since for a simplicial onto map $\varphi : K \to G$ we have $1 \leq |G| \leq |K|$, it follows that $|G|, |G'|, \dots, |G^{(k)}| \leq k$ (recall that we define φ' to be an onto map). This yields that one (and then k-th) of the derivatives $\varphi, \dots, \varphi^{(k)}$ is a standard winding of a nonzero degree, because otherwise we obtain $1 + k \leq |G| + k \leq |G^{(k)}| \leq k$. □

Now let us give the proposed construction of the embedding $G'_\varphi \to \mathbb{R}^2$. It is more convenient for us to consider *thickenings* of the graphs rather than embeddings of the graphs into the plane. Then the proposed construction is equivalent to the construction of *the derivative of a thickening* (Definition of N' below). Further we assume that *a thickening* N of the graph G in the plane (i. e., a regular neighbourhood of $G \subset \mathbb{R}^2$) is fixed. We also assume that a handle decomposition (denoted by S)

$$N = \bigcup_{x \in \text{vertex set of } G} N_x \cup \bigcup_{a \in \text{edge set of } G} N_{(a)}$$

corresponding to the graph G is also fixed, where N_x are 2-discs and $N_{(a)}$ are joining them strips. Denote by N_a the restriction $N_x \cup N_{(a)} \cup N_y$ of N to an edge $a = xy$. Actually, we do not use the planarity of N in our proofs,

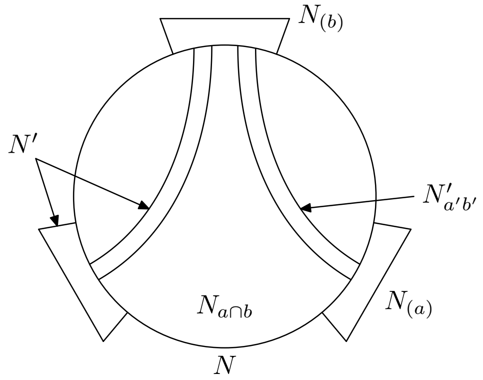

FIGURE 2. Derivative of a thickening

the thickening N can be assumed to be just orientable (orientability is needed for Example 1.1). Let us state the definition of the derivative N' of a thickening N. This thickening N' depends on the simplicial map $\varphi : K \to G \subset N$ and is well-defined only if φ does not contain transversal self-intersections. Moreover, for an arbitrary K we must also assume that there are no pairs of arcs $i, j \subset K$ (not necessarily disjoint!) such that the intersection $\varphi i \cap \varphi j$ is transversal.

Definition (Definition of N', see Fig. 2)**.** Let $\varphi : K \to G \subset N$ be a simplicial map such that for any pair of arcs $i, j \subset K$ the intersection $\varphi i \cap \varphi j$ (maybe empty) is not transversal. Let us construct discs $N'_{a'}$ for each vertex $a' \in G'$ and strips $N'_{(a'b')}$ for each edge $a'b' \subset G'$. Then N' together with its handle decomposition S' is defined by the formula $N' = \bigcup N'_{a'} \cup \bigcup N'_{(a'b')}$. Here we take $N'_{a'} = N_{(a)}$ for each edge $a \subset G$. For each pair $a, b \subset G$ of adjacent edges such that $(\varphi')^{-1}(a'b') \neq \emptyset$ we join the two discs $N'_{a'}$ and $N'_{b'}$ by a narrow strip $N'_{(a'b')}$ in $N_{a\cap b}$. Since the intersection of arcs $a \cup b$ and $c \cup d$ is not transversal for any pair of adjacent edges $c, d \subset K$, it follows that we can choose the strips $N'_{(a'b')}$ so that they do not intersect for distinct $a'b'$.

This definition can also be considered as a construction of an embedding $N' \to N$, and also $G'_\varphi \to \mathbb{R}^2$. Note that S' and the topological type of N' do not depend on the choice of the strips $N'_{(a'b')}$ in our definition. The alternative definition of the derivative thickening $D(N)$ in [6] does not depend also on the map φ. The thickening N' of our paper means the subthickening of $D(N)$ of [6], corresponding to the subgraph $G'_\varphi \subset G'$.

Clearly, for investigation of approximability by embeddings of simplicial maps $K \to G \subset \mathbb{R}^2$ it suffices to consider only the approximations $f : K \to N$. Now we are going to reduce the problem of approximability by embeddings of a given map to the problem of existance of an embedding close to it in some sense (S-close to it).

Definition (Definition of an S-approximation, cf. [6])**.** A map $f : K \to N$ is *an S-approximation* of the map φ, or f is *S-close* to φ, if the following conditions hold:

(1) $fx \subset N_{\varphi x}$ for each vertex or edge x of K
(2) $x \cap f^{-1}N_{(\varphi x)}$ is connected for each edge x of K with nondegenerate φx.

Proposition 2.9 in [6] asserts that the map $\varphi : K \to G$ is approximable by embeddings if and only if there is an embedding $f : K \to N$, S-close to φ.

A PL map $\varphi : K \to N$ is *degenerate*, if φc is a point for some edge $c \subset K$. Now let us prove the following easy Contracting Edge Proposition 2.5 that in some sense allows us to assume that in 2.1 and 2.2 the map φ is nondegenerate.

Proposition 2.5 (Contracting Edge Proposition)**.** *Let $\varphi : K \to G$ be a simplicial map such that φc is a point for some edge $c \subset K$. Let K/c be the graph obtained from K by contracting the edge c, and let $\varphi/c : K/c \to G$ be the corresponding map. Then*

D) $K'_{\varphi/c} = K'_\varphi$, $G'_\varphi = G'_{\varphi/c}$ and $(\varphi/c)' = \varphi'$.
A) for $K \cong S^1$ or $K \cong I$ the map φ/c is approximable by embeddings if and only if φ is approximable by embeddings.
K) for an arbitrary K if φ is approximable by embeddings, then φ/c is approximable by embeddings.
V) If φ is approximable by mod 2*-embeddings, then φ/c is approximable by* mod 2*-embeddings.*

Proof of 2.5. D) is obvious.

A) Let us prove the direct implication. Let $f : K/c \to N$ be an embedding, S-close to φ/c. Let $a \subset K$ be an edge adjacent to c (if c is a connected component of K, then the proposition is obvious). Add a new vertex to the edge a of the graph K/c (Fig. 3.a). Since $K \cong S^1$ or $K \cong I$, it follows that the obtained graph is isomorphic to K and the embedding $f : K \to N$ is the required. The reverse implication is a specific case of statement K).

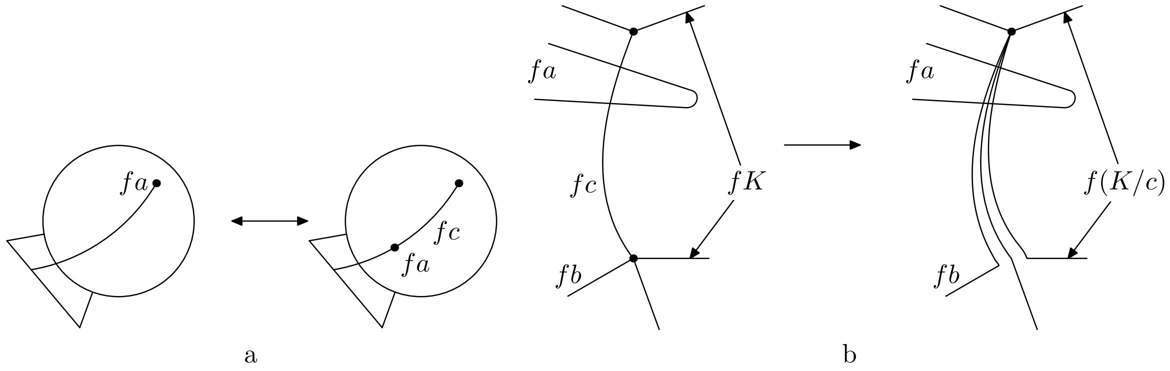

FIGURE 3. Moves for degenerate maps

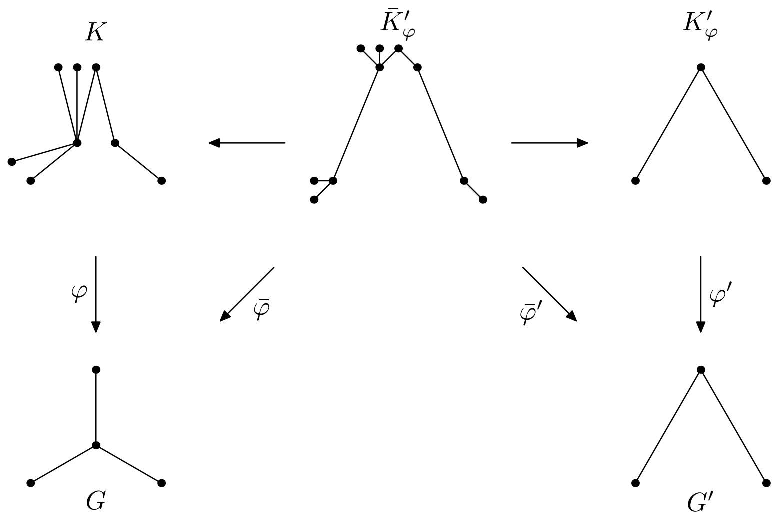

FIGURE 4. Semi-derivatives of a simplicial map

K) Let $f : K \to N$ be an embedding, S-close to φ. Make the move shown in Fig. 3.b. We obtain an embedding $\bar{f} : K/c \to N$, S-close to φ/c.

V) Let f be a mod 2-embedding, S-close to φ. Make the move shown in Fig 3.b. We obtain an S-close to φ/c map $\bar{f} : K/c \to N$. It suffices to prove that $|\bar{f}a \cap \bar{f}b| = 0 \pmod 2$ for each pair of disjoint edges $a, b \subset (K/c)$. Indeed, both a and b are also edges of K, and at least one of them is not adjacent to c (because a and b are disjoint in K/c). If neither a nor b is adjacent to c, then $|\bar{f}a \cap \bar{f}b| = |fa \cap fb| = 0 \pmod 2$. If, for example, $b \in K$ is adjacent to c and a is not adjacent to c, then $|\bar{f}a \cap \bar{f}b| = |fa \cap fb| + |fa \cap fc| = 0 \pmod 2$, that proves the proposition. □

Degenerate maps appear in our proof of 2.1 and 2.2 even if the map $\varphi : K \to G$ is nondegenerate. We are going to construct a graph $\bar{K}'_\varphi$ and a pair of (degenerate) simplicial maps $G \xleftarrow{\bar{\varphi}} \bar{K}'_\varphi \xrightarrow{\bar{\varphi}'} G'$ that can be obtained from φ and φ' respectively by the operation from Contracting Edge Proposition 2.5 (this is true under some assumptions on φ, we present the details below). Together with the construction of the embedding $N' \to N$ (see Definition of N' above) this immediately proves 2.1 (Fig. 4, 5, 6).

Definition (Definition of $\bar{\varphi}$ and $\bar{\varphi}'$, see Fig. 4)**.** Suppose that the map φ is nondegenerate and K does not have vertices of degree 0. Take the disjoint union of all φ-components of K (see Definition of φ'). Join by an edge any two vertices belonging to distinct φ-components and corresponding to the same vertex of K. Denote the obtained *semi-derivative* graph by $\bar{K}'_\varphi$. Thus a φ-component $\alpha \subset K$ is also a subgraph of $\bar{K}'_\varphi$ denoted by $\bar{\alpha}'$. Further we identify the points of α and $\bar{\alpha}'$. Let the simplicial maps $\bar{\varphi}$ and $\bar{\varphi}'$ be the evident projections $\bar{K}'_\varphi \to G$ and $\bar{K}'_\varphi \to G'$ respectively, defined on the vertex sets by $\bar{\varphi}x = \varphi x$ and $\bar{\varphi}'x = (\varphi\alpha)'$, where the vertex $x \in \bar{K}'_\varphi$ belongs to the φ-component $\bar{\alpha}'$.

Proof of 2.1. By Contracting Edge Proposition 2.5.D,A the map φ can be assumed to be nondegenerate. We also may assume that K does not have vertices of degree 0. It can be easily checked that φ and φ' can be obtained from $\bar{\varphi}$ and certain restriction of $\bar{\varphi}'$ respectively by the operation from Contracting Edge Proposition 2.5. If any two φ-components have at most one common point, then φ' can be obtained from $\bar{\varphi}$ itself in this way. But for $K \cong S^1$ this asumption is

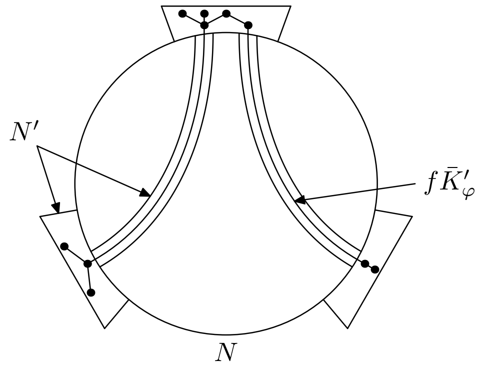

FIGURE 5. Construction of the S-approximation

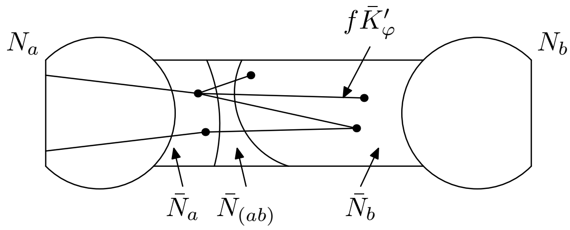

FIGURE 6. Construction of the handle decomposition

not satisfied only if K has two φ-components. Evidently, the map φ is approximable by embeddings in this case. So it suffices to prove that

(*) if $\bar{\varphi}'$ is approximable by embeddings, then $\bar{\varphi}$ is approximable by embeddings.

We prove $(*)$ for an arbitrary graph K. If $\bar{\varphi}'$ is approximable by embeddings, then there is a S'-close to $\bar{\varphi}'$ embedding $\bar{K}'_\varphi \to N'$. Define the embedding $f : \bar{K}'_\varphi \to N$ to be the composition of this embedding and the embedding $N' \to N$ constructed in Definition of N' (Fig. 5, where this construction is applied to the map φ from Fig. 4). Clearly, there exists a new handle decomposition $N = \bigcup \bar{N}_a \cup \bigcup \bar{N}_{(ab)}$, denoted by $\bar{S}$, such that f is an $\bar{S}$-approximation of $\bar{\varphi}$ (Fig. 6, cf. [6, Proposition 4.9], or see some generalization of the decomposition, constructed in the proof of Lemma 4.5.D.) Then $f : \bar{K}'_\varphi \to \bar{N}$ (where $\bar{N}$ is N with the new handle decomposition $\bar{S}$) is an embedding, $\bar{S}$-close to $\bar{\varphi}$, that proves the lemma. $\square$

The same idea is used in the proof of Lemma 2.2.A,V. We take a general position map $f : \bar{K}'_\varphi \to N$, S-close to $\bar{\varphi}$, and construct its *semi-derivative* $\bar{f}' : \bar{K}'_\varphi \to N'$, S-close to $\bar{\varphi}'$ (Fig. 7). Then we prove that if f is an embedding, then $\bar{f}'$ is also an embedding (Fig. 8).

Definition (Definition of $\bar{f}'$, see Fig. 7, where this construction is applied to the map φ from Fig. 4)**.** Let K be a graph without vertices of degree 0. Let $\varphi : K \to G \subset N$ be a nondegenerate simplicial map without transversal self-intersections. Let $f : K \to N$ be an S-approximation of φ. Then *the semi-derivative* S'-approximation $\bar{f}' : \bar{K}'_\varphi \to N'$ is constructed as follows. For each edge $a \subset G$ fix a homeomorphism $h_a : N_a \to N'_{a'}$ such that for each edge b adjacent to a we have $h_a(N_a \cap N_{(b)}) \subset N'_{(a'b')}$. Define $\bar{f}'$ on each φ-component $\bar{\alpha}' \subset \bar{K}'_\varphi$ by the formula $\bar{f}'|_{\bar{\alpha}'} = h_{\varphi\alpha} f|_\alpha$ Now let us define $\bar{f}'$ on each edge $xy \subset \bar{K}'_\varphi$ joining two distinct φ-components $\bar{X}'$ and $\bar{Y}'$. Take an edge $a \subset \bar{X}'$ containing x. Identify $\bar{X}'$ and X (see Definition of $\bar{\varphi}$ and $\bar{\varphi}'$). Then a is also an edge of K and x is also a vertex of K. Denote by $\bar{x}$ the arc $a \cap f^{-1}N_{\varphi x}$. Define the arc $\bar{y}$ analogously. Decompose the edge xy into three segments xx_1, x_1y_1 and y_1y. Let $\bar{f}'$ homeomorphically map xx_1 onto $h_{\varphi X} f\bar{y}$, y_1y onto $h_{\varphi Y} f\bar{x}$, and x_1y_1 onto the rectilinear segment in $N'_{(\varphi X\, \varphi Y)}$ joining the points $\bar{f}'x_1$ and $\bar{f}'y_1$. Thus the map $\bar{f}' : \bar{K}'_\varphi \to N'$ is constructed.

Note that if f is an embedding then there is a simpler alternative construction of $\bar{f}'$, in some sense reverse to the construction from the proof of Lemma 2.1. But this alternative construction is useless in the proof of Lemma 2.2.V, so we do not present it in the paper. We prove 2.2.A,V only in case when the derivative N' is well-defined, i. e. K does not contain pairs of arcs i, j such that $\varphi i \cap \varphi j$ is transversal. This is sufficient for the proof of Theorem 1.3 and 1.5. In general case the proof is completely analogous, but one should use the definition of derivative $D(N)$ from [6].

Proof of 2.2.A. By Contracting Edge Proposition 2.5.K we may assume that φ is nondegenerate. Take an embedding $f : K \to N$, S-close to φ. Then it suffices to show that the map $\bar{f}'$ (see Definition of $\bar{f}'$) is an embedding.

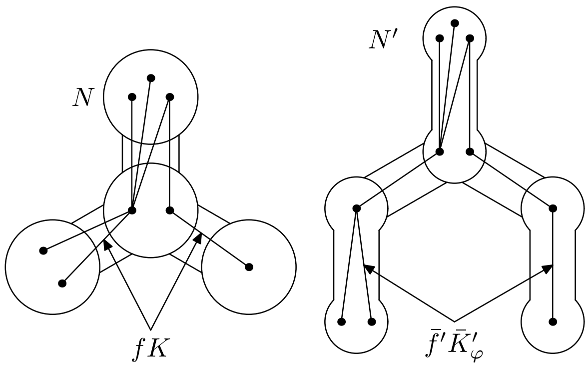

FIGURE 7. Semiderivative of an S-approximation

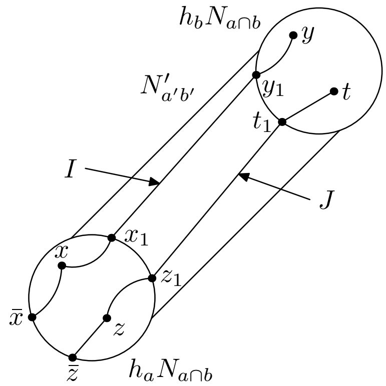

FIGURE 8. Counting the number of crossings

Consider a pair of distinct edges xy, zt of K'_φ. Denote the set $\bar{f}'(xy) \cap \bar{f}'(zt)$ by i. It suffices to show that $i = \bar{f}'(xy \cap zt)$. Denote by $a' = \bar{\varphi}'x$, $b' = \bar{\varphi}'y$, $c' = \bar{\varphi}'z$ and $d' = \bar{\varphi}'t$. Without loss of generality we have the following 3 cases.

1) a', b', c' and d' are pairwise distinct. Since $\bar{f}'$ is an S'-approximation, it follows that $\bar{f}'xy \subset N'_{a'b'}$ and $\bar{f}'zt \subset N'_{c'd'}$, hence $i = \emptyset$.

2) ($a' = c'$ and $b' \neq d'$) or ($a' = b' = c' = d'$). Then $i \subset N'_{a'}$, hence $i = h_a(f\bar{x} \cap f\bar{z})$ (see the definition of h_a and $\bar{x}$ in Definition of $\bar{f}'$, define $\bar{z}$ analogously to $\bar{x}$.) If $y \neq t$, then $\bar{x}$ and $\bar{z}$ are disjoint, so $f\bar{x} \cap f\bar{z} = \emptyset$ and $i = \emptyset$. If $y = t$, then $i = h_a(fy) = \bar{f}'(xy \cap zt)$.

3) $a' = c'$, $b' = d'$ and $a' \neq b'$. In this case both xy and zt join the vertices of distinct φ-components. Let us prove that xy and zt are disjoint. For example, assume that $y = t$. Then all the vertices x, y, z and t of $\bar{K}'_\varphi$ correspond to the same vertex of K denoted by w. Denote by X and Z the φ-components of $\varphi^{-1}a = \varphi^{-1}c$ such that $x \in \bar{X}'$ and $z \in \bar{Z}'$. So the φ-components X and Z have a common point w, hence $X = Z$. So $x, z \in \bar{X}' = \bar{Z}'$ correspond to the same vertex w, hence $x = z$. We obtain $y = t$ and $x = z$, then by the construction of $\bar{K}'_\varphi$ we get $xy = zt$, that contradicts to the choice of these edges. So xy and zt are disjoint.

Let us show that in case (3) $|i| = 0 \pmod 2$. Omit $\bar{f}'$ from the notation of $\bar{f}'$-images. Note that the homeomorphism $h_a \circ h_b^{-1}$ maps y_1y and t_1t onto $\bar{x}$ and $\bar{z}$ respectively (Fig. 8). First this implies that $|i| = |I \cap J|$, where $I = \bar{x} \cup xy_1$ and $J = \bar{z} \cup zt_1$. Secondly this implies that the two pairs of points ∂I and ∂J are not linked in $\partial(h_a N_{a\cap b} \cup N'_{(a'b')})$. Since $I, J \subset h_a N_{a\cap b} \cup N_{(a'b')}$, it follows that $|i| = |I \cap J| = 0 \pmod 2$. So it remains to prove that $|I \cap J| \leq 1$, then $I \cap J = \emptyset$. This follows from

$$\bar{x} \cap \bar{z} = h_a(f\bar{x} \cap f\bar{z}) = \emptyset \qquad xx_1 \cap zz_1 = h_a(f\bar{y} \cap f\bar{t}) = \emptyset \qquad \text{and} \qquad |x_1y_1 \cap z_1t_1| \leq 1,$$

because x_1y_1 and z_1t_1 are rectilinear segments in $N_{(a'b')}$. This completes the proof of the lemma. □

Proof of 2.2.V. By Contracting Edge Proposition 2.5.V it suffices to prove that if $f : \bar{K}'_\varphi \to N$ is a mod 2-embedding, S-close to φ, then the semi-derivative $\bar{f}'$ is also a mod 2-embedding.

Take a pair of disjoint edges xy, zt of $\bar{\varphi}'$ and consider the three cases from the proof of Lemma 2.2.A. Case 1) is trivial. In case 2) we get $f(xy) \cap f(zt) \subset N_a$, hence $|i| = |h_a(f\bar{x} \cap f\bar{z})| = |h_a(f(xy) \cap f(zt))| = |f(xy) \cap f(zt)| = 0 \pmod 2$. In the proof of 2.2.A it is shown that in case 3) $|i| = 0 \pmod 2$, thus Lemma 2.2.V is proved. □

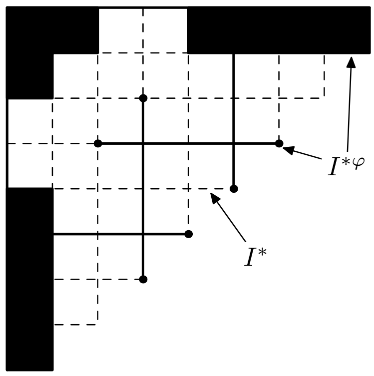

FIGURE 9. The Van Kampen obstruction

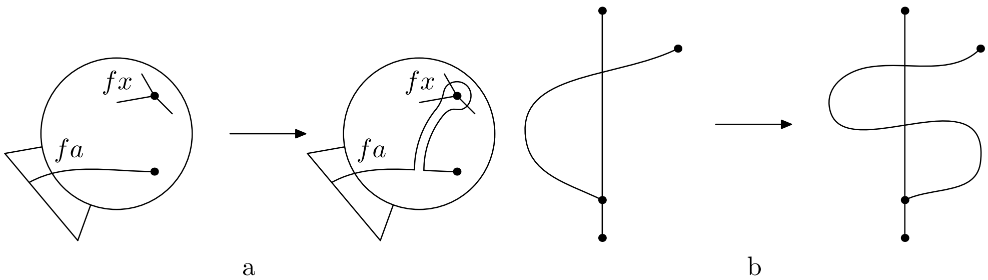

FIGURE 10. "The Reidemeister move"

3. The van Kampen obstruction

The van Kampen obstruction was invented by van Kampen in studies of embeddability of polyhedra in $\mathbb{R}^{2n}$ [2, 3, 4, 7, 8]. Let us give the definition of the van Kampen obstruction to approximability by embeddings of simplicial paths. Our construction is more visual than that in the problem of embeddability. Let $\varphi : I \to \mathbb{R}^2$ be a simplicial path (in Fig. 9 the constuction below is applied to the path from Fig. 1). Denote by $x_1, \dots, x_k$ the vertices of I in the order along I, and denote the edge $x_i x_{i+1}$ by i. Let $I^* = \bigcup\limits_{i<j-1} i \times j$ be *the deleted product* of I. Paint red the edges $x_i \times j$, $j \times x_i$, and the cells $i \times j$ of I^* such that $\varphi x_i \cap \varphi j = \emptyset$, $\varphi i \cap \varphi j = \emptyset$, and denote by $I^{*\varphi}$ the red set. Take a general position map $f : I \to \mathbb{R}^2$, sufficently close to φ. To each cell $i \times j$ of "the table" I^* put the number $v_f(i \times j) = |fi \cap fj| \pmod 2$. Decompose I^* along the red edges, let $C_1, C_2, \dots, C_n$ be all the obtained components such that $\partial C_k \cap \partial I^* \subset I^{*\varphi}$. Denote $v_f(C_k) = \sum\limits_{i \times j \subset C_k} v_f(i \times j)$. *The van Kampen obstruction* (with $\mathbb{Z}_2$-coefficients) for approximability by embeddings is the vector $v(\varphi) = (\, v_f(C_1), v_f(C_2), \dots, v_f(C_n)\,)$.

It can be shown easily that $v(\varphi)$ does not depend on the choice of f [8], thus $v(\varphi) = 0$ is a necessary condition for approximability by embeddings. It is easy to check that $v(\varphi) \neq 0$ for a PL path $\varphi : I \to \mathbb{R}^2$ containing a transversal self-intersection. Thus Corollary 1.4.V follows from 1.3, 2.2.V and 3.1.

Proposition 3.1. *The van Kampen obstruction $v(\varphi) = 0$ if and only if there is an S-close to φ general position* mod 2*-embedding.*

Proof of 3.1. The inverse implication of the proposition is obvious. The proof of the direct implication follows the idea of [4]. We are going to use the cohomogical formulation of the van Kampen obstruction (see the paragraph before Proposition 3.2 below for details). Let $f : K \to N$ be any general position S-approximation of φ. The 'Reidemeister move' shown in Fig. 10.a adds to v_f the coboundary $\delta[x \times a]$ of the elementary cochain from $B^2(\tilde{K})$. Since $v(\varphi) = 0$, it follows that using some such 'moves' we can obtain a map $f : K \to N$ such that $v_f = 0$. Then f is the required mod 2-embedding, because $v_f = 0$ yields that $|fa \cap fb| = 0 \pmod 2$ for any pair of disjoint edges a, b of K. $\square$

Now we are going to prove that the conditions 1.4.V and 1.4.D are equivalent (Proposition 3.2). We are going to replace $\mathbb{Z}_2$-coefficients in the van Kampen obstruction by $\mathbb{Z}$-coefficients, so Proposition 3.2 implies only that $1.4.D \implies 1.4.V$, but this is sufficient for the proof of Corollary 1.4. We prove Proposition 3.2 in the most general situation, so we need some more definitions.

Let K be an n-polyhedron with a fixed triangulation. Let $\varphi : K \to G \subset \mathbb{R}^{2n}$ be a simplicial map. Denote by σ and τ any n-dimensional simplices of this triangulation of K. By *the deleted product* of K we mean the set $\tilde{K} = \bigcup\{\, \sigma \times \tau : \sigma \cap \tau = \emptyset \,\}$. Fix the natural orientation of each cell $\sigma \times \tau$ (a positive basis of $\sigma \times \tau$ consists of the vectors $e_1, \dots, e_{2n}$, where $e_1, \dots, e_n$ form a positive basis of σ and $e_{n+1}, \dots, e_{2n}$ form a positive basis of τ). Let

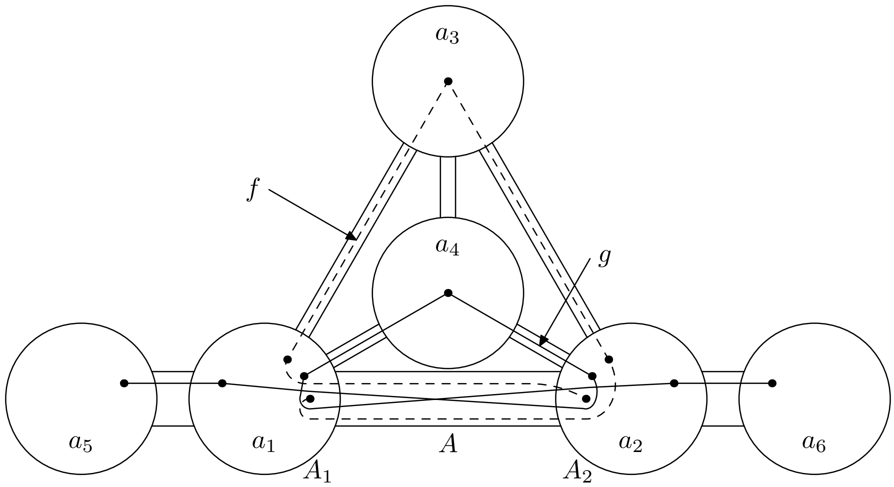

FIGURE 11. A pair of maps not approximable by link maps

$K^* = \tilde{K}/\mathbb{Z}_2$ be the factor under antipodal $\mathbb{Z}_2$-action. Let $\tilde{K}^\varphi \subset \tilde{K}$ be the subset $\tilde{K}^\varphi = \{\, \sigma \times \tau : \varphi\sigma \cap \varphi\tau = \emptyset \,\}$ and let $K^{*\varphi} = \tilde{K}^\varphi/\mathbb{Z}_2$. For a general position map $f : K \to \mathbb{R}^{2n}$ close to φ define a cochain $v_f \in C^n(K^*, K^{*\varphi}; \mathbb{Z})$ by the formula $v_f(\sigma \times \tau) = f\sigma \cap f\tau$. This cochain is well-defined, because $f\sigma \cap f\tau = (-1)^n f\tau \cap f\sigma$ and our $\mathbb{Z}_2$-action maps $\sigma \times \tau$ to $(-1)^n \tau \times \sigma$. The class $v(\varphi) = [v_f] \in H^n(K^*, K^{*\varphi}; \mathbb{Z})$ of this cochain does not depend on the map f and is called *the van Kampen obstruction* to approximability of φ by embeddings. We say that the map $\varphi : K \to G \subset \mathbb{R}^{2n}$ satisfies *the deleted product property* if the map $\tilde{\varphi} : \tilde{K}^\varphi \to S^{2n-1}$ given by the formula $\tilde{\varphi}(x, y) = \frac{\varphi x - \varphi y}{\|\varphi x - \varphi y\|}$ extends to an equivariant map $\tilde{K} \to S^{2n-1}$. Evidently, this definition of the deleted product property is equivalent to 1.4.D for $n = 1$ and $K \cong I$.

Proposition 3.2. *A PL map $\varphi : K^n \to \mathbb{R}^{2n}$ satisfies the deleted product property if and only if the van Kampen obstruction (with $\mathbb{Z}$-coefficients) $v(\varphi) = 0$.*

Proof of 3.2. We are going to show that the van Kampen obstruction is a complete obstruction to an equivariant extension of $\tilde{\varphi} : \tilde{K}^\varphi \to S^{2n-1}$ to a map $\tilde{K} \to S^{2n-1}$.

Take a general position map $f : K \to \mathbb{R}^{2n}$ close to φ and define the equivariant map $\tilde{f} : \tilde{K}^\varphi \cup \operatorname{sk}^{2n-1} \tilde{K} \to S^{2n-1}$ by the formula $\tilde{f}(x, y) = \frac{fx - fy}{|fx - fy|}$. By general position it follows that $\tilde{f}$ is well-defined. Since f is close to φ, it follows that $\tilde{f}|_{\tilde{K}^\varphi}$ is homotopic to $\tilde{\varphi}$. Evidently, then $\tilde{\varphi}$ extends to an equivariant map $\tilde{K} \to S^{2n-1}$ if and only if $\tilde{f}|_{\tilde{K}^\varphi}$ extends to an equivariant map $\tilde{K} \to S^{2n-1}$.

Consider a cell $\sigma \times \tau \subset \tilde{K} - \tilde{K}^\varphi$, where $\sigma, \tau \subset K$ are n-dimensional cells. The map $\tilde{f}$ extends to $\sigma \times \tau$ if and only if $\deg \tilde{f}|\partial(\sigma \times \tau) = 0$. If $\tilde{f}$ extends to $\sigma \times \tau$ then it extends also to $\tau \times \sigma$ in equivariant way, because $\tilde{f}$ is equivariant. One can see that $\deg \tilde{f}|\partial(\sigma \times \tau) = f\sigma \cap f\tau = v_f(\sigma \times \tau)$. So the map $\tilde{f}$ extends to an equivariant map $\tilde{K} \to S^{n-1}$ if and only if $v_f = 0$.

Now let $g : \tilde{K}^\varphi \cup \operatorname{sk}^{2n-1} \tilde{K} \to S^{2n-1}$ be an equivariant map such that $gx = \tilde{f}x$ for each $x \in \tilde{K}^\varphi \cup \operatorname{sk}^{2n-2} \tilde{K}$. Define the cochain $v_g \in C^{2n}(K^*, K^{*\varphi}; \mathbb{Z})$ by the formula $v_g(\sigma) = \deg g|\partial\sigma$ for each $2n$-dimensional cell σ. Let $\sigma \subset \tilde{K} - \tilde{K}^\varphi$ be a cell of dimension $2n - 1$. Take a disjoint union $\sigma \sqcup \sigma'$ of two copies of σ and paste σ to σ' by $\partial\sigma = \partial\sigma'$. Let d_σ be a map of the obtained $(2n-1)$-sphere to S^{2n-1} given by the formula $d_\sigma x = fx$ for $x \in \sigma$ and $d_\sigma x = gx$ for $x \in \sigma'$. Define the cochain $v_{fg} \in C^{2n-1}(K^*, K^{*\varphi}; \mathbb{Z})$ by the formula $v_{fg}(\sigma) = \deg d_\sigma$ (we fix the orientation of $\sigma \cup \sigma'$ restricted to the positive orientation of σ'). Then, clearly $v_g - v_f = \delta v_{fg}$.

The obtained formula implies that the cohomological class $[v_g]$ does not depend on the choice of an equivariant map $g : \tilde{K}^\varphi \cup \operatorname{sk}^{2n-1} \tilde{K} \to S^{2n-1}$ and coincides with the van Kampen obstruction $v(\varphi)$ (with $\mathbb{Z}$-coefficients). This proves that the condition $v(\varphi) = 0$ in the proposition is necessary. The obtained formula and the construction of v_{fg} above shows that if $v(\varphi) = 0$ then $v_g = 0$ for some $g : \tilde{K}^\varphi \cup \operatorname{sk}^{2n-1} \tilde{K} \to S^{2n-1}$, hence $\tilde{f}|_{\tilde{K}^\varphi}$ extends to an eguivariant map $\tilde{K} \to S^{2n-1}$. So the proposition is proved. $\square$

Example 3.3. (cf. [15, 1]) There exists a pair of PL paths $\varphi : I \to \mathbb{R}^2$, $\psi : I \to \mathbb{R}^2$ (Fig. 11, where a pair of paths f, g, close to them, is shown), not approximable by link maps (i. e., maps with disjoint images) and such that:

V) *The van Kampen obstruction $v(\varphi, \psi) = 0$.*

D) The map $\Phi : \{\, (x, y) \in I \times I \,|\, \varphi x \neq \psi y \,\} \to S^1$ given by $\Phi(x, y) = \frac{\varphi x - \psi y}{\|\varphi x - \psi y\|}$ homotopically extends to a map $I \times I \to S^1$.

I) The pair φ', ψ' is approximable by link maps.

Proof of 3.3. Let $K, L \cong I$ are the graphs with the vertices $k_1, \dots, k_5$ and $l_1, \dots, l_7$, let G be the graph with vertices $a_1, \dots, a_6$ and edges a_1a_2, a_1a_3, a_1a_4, a_1a_5, a_2a_3, a_2a_4 and a_2a_6. The required simplicial maps φ, ψ are given by the formulae $\varphi k_1 = a_1$, $\varphi k_2 = a_2$, $\varphi k_3 = a_3$, $\varphi k_4 = a_1$, $\varphi k_5 = a_2$ and $\psi l_1 = a_5$, $\psi l_2 = a_1$, $\psi l_3 = a_2$, $\psi l_4 = a_4$, $\psi l_5 = a_1$, $\psi l_6 = a_2$, $\psi l_7 = a_6$. Consider the pair of S-approximations f and g of φ and ψ respectively shown in Fig. 11. One can see that $|fi \cap gj| = 0 \pmod 2$ for any pair of edges $i \subset K, j \subset L$. This implies both 3.3.V and 3.3.D (it is shown analogously to the proof of 1.4, see also Proposition 3.1). The proof of 3.3.I is a direct calculation. Let us prove that the pair φ, ψ is not approximable by link maps. Assume the converse. Let $K_{13}, K_{35} \subset K$ and $L_{14}, L_{47} \subset L$ be the arcs between the points k_1 and k_3, k_3 and k_5, l_1 and l_4, l_4 and l_7 respectively. Take a small neighbourhood of $\varphi K \cup \psi L$ in the plane and fix its handle decomposition S. Denote by A_1, A_2 and A the discs of S corresponding to the vertices a_1, a_2 and to the edge a_1a_2 respectively. By the analogue of the Minc Proposition (see the paragraph after Definition of an S-approximation in §2) there are two S-approximations f, g of φ and ψ respectively, having disjoint images. Since $fK_{13} \cap gL = \emptyset$, it follows that the pairs of points $gL_{14} \cap \partial(A_1 \cup A)$ and $gL_{47} \cap \partial(A_1 \cup A)$ are not linked in $\partial(A_1 \cup A)$. Analogously, $gL_{14} \cap \partial A_2$ and $gL_{47} \cap \partial A_2$ are not linked in ∂A_2. So $gL_{14} \cap \partial(A_1 \cup A_2 \cup A)$ and $gL_{47} \cap \partial(A_1 \cup A_2 \cup A)$ are not linked in $\partial(A_1 \cup A_2 \cup A)$. Then g cannot be an S-approximation of ψ. This contradiction proves that φ and ψ are not approximable by link maps. $\square$

4. Variations

The following Simple-minded Criterion 4.1 for approximability by embeddings gives an algorithm of checking whether a given nondegenerate map is approximable by embeddings (another algorithm is given in [13]).

Proposition 4.1. *Simple-minded Criterion 4.1 Let $\varphi : K \to G \subset \mathbb{R}^2$ be a nondegenerate simplicial map of a graph K, i. e. for each edge a of K the image φa is not a vertex. Replace each edge $a \subset G$ by i close multiple edges in $\mathbb{R}^2$, if $\varphi^{-1}a$ consists of i edges. Denote by $\bar{G} \subset \mathbb{R}^2$ the obtained graph and by $\pi : \bar{G} \to G$ the evidently defined projection. The map φ is approximable by embeddings if and only if there exists an onto map $\bar{\varphi} : K \to \bar{G}$ without transversal self-intersections and such that $\pi \circ \bar{\varphi} = \varphi$.*

The proof is trivial (we do not present the details since we do not use this criterion). There exists a purely combinatorial proof of Theorem 1.3, based on Criterion 4.1. Criterion 4.1 and all the other our previous results remain true, if we replace $\mathbb{R}^2$ by an arbitrary orientable 2-manifold N.

There exists an infinite number of PL maps $\varphi : T \to T \subset \mathbb{R}^2$, where T is letter "T" (*a simple triod*), not approximable by embeddings and such that φ' and any simplicial restriction of φ are approximable by embeddings (for the only embedding $T' \to \mathbb{R}^2$). So there are no criteria like 1.3.I,S for $K \not\cong I, S^1$.

In the rest of the paper we prove Theorem 1.5, generalizing both 1.3 and 1.4. For the proof we need the following Lemma 4.2.T, Lemma 4.5 and Lemma 4.6, analogous to Lemmas 2.3, 2.1 and 2.2 respectively.

To state Lemma. 4.2 we need the following definitions. We shall say that φ *contains a simple triod*, if there is a triod $T \subset K$ with the edges t_1, t_2, t_3 such that the arcs $\varphi t_1, \varphi t_2, \varphi t_3$ have a unique common point. We shall say that φ *identifies triods*, if it contains two disjoint simple triods $T_1, T_2 \subset K$ such that $\varphi T_1 = \varphi T_2$. Note that $v(\varphi) \neq 0$ for a map φ identifying triods. We shall say that an onto map $\varphi : K \to G$ is *a standard winding*, if both K and G are homeomorphic to disjoint unions of circles (may be, $K = G = \emptyset$) and $\varphi|_S$ is a standard winding of a nonzero degree for each circle $S \subset K$. Denote by $\Sigma(\varphi) = \{\, x \in K \,:\, |\varphi^{-1}\varphi x| \geq 2 \,\}$ *the singular set* of the map φ.

Lemma 4.2. *(cf. Lemma 2.3) Let $\varphi : K \to G$ be a simplicial map of a graph K with k vertices.*

T) Suppose that for each $i = 0, \dots, k$ the derivative $\varphi^{(i)}$ does not contain simple triods; then $\varphi^{(k)}$ is a standard winding.

I) Suppose that for each $i = 0, \dots, k$ the derivative $\varphi^{(i)}$ does not identify triods; then $\Sigma(\varphi^{(k)})$ is a disjoint union of circles and $\varphi^{(k)}|_{\Sigma(\varphi^{(k)})}$ is a standard winding.

Proof of 4.2.T. We are going to use the notation from the proof of Lemma 2.3. Let us show that $|K| \geq |K'_\varphi|$ for a simplicial map $\varphi : K \to G$ containing no simple triods. We also prove that $|K| = |K'_\varphi|$ only if K is homeomorphic to a disjoint (maybe empty) union of circles and φ is ultra-nondegenerate. Then it suffices to prove the lemma for this latter specific case. Indeed, since φ does not contain simple triods, it follows that each vertex of the graph K belongs to at most two φ-components of the graph. On the other hand, each φ-component contains an edge, and hence it contains at least two vertices. This yields that the number of vertices of K is greater or equals to the number of φ-components, i. e. $|K| \geq |K'_\varphi|$. We have the equation here if and only if (1) each vertex of K belongs to two φ-components and (2) each φ-component contains exactly two vertices. The condition (2) means that φ is ultra-nondegenerate. But for an ultra-nondegenerate map the condition (1) yields that the degree of each vertex of K is 2, so K is a disjoint union of circles. Now note that for any two components $A, B \subset K$ we have $\varphi' A' \cap \varphi' B' = (\varphi A \cap \varphi B)'$. Since $A, B \cong S^1$ and φ is ultra-nondegenerate, it follows that $\varphi A \cap \varphi B$ is always either empty or a circle or a disjoint union of arcs and points. Moreover, $\varphi A \cap \varphi B$ is a circle if and only if $\varphi A = \varphi B$. If $\varphi A \cap \varphi B$ is either empty or a disjoint union of arcs and points, then $\varphi^{(k)} A$ and $\varphi^{(k)} B$ are disjoint. So the images $\varphi^{(k)} A$ and $\varphi^{(k)} B$ either are disjoint or coincide. By Lemma 2.3 this yields that $\varphi^{(k)}$ is a standard winding. $\square$

We do not use Lemma 4.2.I and prove it after the proof of Theorem 1.5. Lemma 4.2.I may be helpful in the proof of Conjecture 1.6.

In order to prove Theorem 1.5 we need the following extension of the techniques from §2. As in §2, we fix an orientable thickening N of the graph $G \cong S^1$. We also assume that some orientable thickening M of the graph K is fixed. Note that a thickening of a graph is uniquely defined by *a local ordering* of edges around each vertex [6]. We assume that K may contain loops and multiple edges. In this case by *a simplicial map* $\varphi : K \to G$ we mean a continuous map that is linear on each edge of K and such that φx is a vertex of G for each vertex $x \in K$. We assume that the handle decompositions $M = \bigcup M_x \cup \bigcup M_{(a)}$ and $N = \bigcup N_x \cup \bigcup N_{(a)}$ are fixed (in both formulae the first union is over all vertices x and the second — over all edges a). By M_α and N_β we denote the restriction of the thickenings M and N to subgraphs $\alpha \subset K$ and $\beta \subset G$ respectively. By *an S-approximation* of φ (or *S-close to φ* map) we mean a general position map $f : M \to N$ such that for any vertex $x \in K$ or edge $x \subset K$ we have $fM_x \subset N_{\varphi x}$ and for any edge $x \subset K$ with nondegenerate φx the set $M_x \cap f^{-1}N_{(\varphi x)}$ is connected (cf. Definition of S-approximation in §2). If there is an S-close to φ embedding $M \to N$ then we shall say that *φ is approximable by embeddings $M \to N$*. By *a* mod 2*-embedding* we mean a general position map $f : M \to N$ such that $f\,|_{M_x}$ is an embedding for each vertex $x \in K$, $f\,\big|_{M_{(a)}}$ is an immersion for each edge $a \subset K$ and $|fa \cap fb - f(a \cap b)| = 0 \pmod 2$ for any pair of distinct edges $a, b \subset K \subset M$. The last notion appears in the following generalization of Proposition 3.1.

Lemma 4.3. *Let K be a graph such that the degree of each vertex of K is at most 3. Let $\varphi : K \to G \subset \mathbb{R}^2$ be a simplicial map such that $v(\varphi) = 0$. Then there exist a* mod 2*-embedding $M \to N$, S-close to φ, for some thickenings M and $N \subset \mathbb{R}^2$ of the graphs K and G respectively.*

Proof of 4.3. [10] Let N be a regular neighbourhood of G in $\mathbb{R}^2$. Let $f : K \to N$ be the map given by Proposition 3.1. Since the degree of each vertex of K is at most 3, it follows that we can remove intersections of adjacent edges, using the moves shown in Fig. 10.b. The obtained general position map $K \to N$ uniquely defines a thickening M of K and extends to the required mod 2-embedding, S-close to φ. □

In [10] the move shown in Fig. 10.b is assumed to work for vertices of any degree, that is not right. The degree restriction in Theorem 1.5 is used only in this step of the proof.

Now we are going to construct *the derivative* M'_φ of the thickening M. This derivative is well-defined only under the following conditions on φ, M and N. We shall say that $\varphi : K \to G$ is *locally approximable by embeddings*, if for each vertex $x \in G$ there exists an S-approximation $f : M \to N$ of φ such that $f\,\big|_{f^{-1}N_x}$ is an embedding. The following lemma asserts that the thickenings M and N given by Lemma 4.3 satisfy these conditions.

Lemma 4.4 (Local Approximation Lemma)**.** *If there is a* mod 2*-embedding $M \to N$ which is S-close to the map $\varphi : K \to G$, then φ is locally approximable by embeddings $M \to N$.*

Proof of 4.4. Let $f : M \to N$ be an S-close to φ mod 2-embedding and let x be a vertex of the graph G. Modify f in a small neighbourhood of ∂N_x to obtain a map f such that for each edge $a \ni x$ the set $f^{-1}(N_{(a)} \cap N_x)$ is a single point P_a. Attach a ring R to the disc N_x along the circle ∂N_x. Let Q be the 2-polyhedron obtained from $M_{\varphi^{-1}x} \cup (K \cap f^{-1}N_x)$ by identifying the points $f^{-1}P_a$ for each $a \ni x$. Identify each point $f^{-1}P_a \in Q$ with P_a. Attach the ring R to Q by the inclusions $P_a \subset R$. Let $g : Q \cup R \to N_x \cup R$ be the map given by the formulae $gy = fy$ for $y \in Q$ and $gy = y$ for $y \in R$. Clearly, g is *a* mod 2*-embedding*, i. e. there exists a triangulation of $Q \cup R$ such that $|ga \cap gb| = 0 \pmod 2$ for each pair of disjoint edges a, b of this triangulation. This yields that $Q \cup R$ contains neither Kuratowsky graph K_5 nor $K_{3,3}$. Clearly, $Q \cup R$ contains neither S^2 nor the cone over $S^1 \sqcup D^0$. By the well-known 2-polyhedron planarity criterion $Q \cup R$ is planar. So the embedding $R \subset N_x \cup R$ extends to an embedding $h : Q \cup R \to N_x \cup R$. Clearly, $h\,|_Q : Q \to N_x$ can be modified to an embedding $f^{-1}N_x \to N_x$, that extends to the required S-approximation of φ. So φ is locally approximable by embeddings. □

The next step in the construction of M'_φ is like Contracting Edge Proposition 2.5. We use a reduction to the case of a nondegenerate map φ and then define the semi-derivative thickening.

Definition (Definition of φ^c)**.** First let us define the graph K^c_φ, which is the domain of φ^c. A 0*-component* of K is any connected component of $\varphi^{-1}x$ for a vertex $x \in G$. The vertex set of the graph K^c_φ is in 1-1 correspondence with the set of all 0-components. Denote by α^c the vertex corresponding to a 0-component $\alpha \subset K$. The vertices α^c and β^c are joined by an edge in K^c_φ if and only if K contains an edge with two ends belonging to α and β respectively. The map $\varphi^c : K^c_\varphi \to G$ is a simplicial map given by the formula $\varphi^c\alpha^c = \varphi\alpha$.

Definition (Definition of M^c)**.** Let the map $\varphi : K \to G$ be locally approximable by embeddings. The thickening M^c_φ and its handle decomposition are defined as follows. For each 0-component $\alpha \subset K$ choose a maximal tree $T \subset \alpha$. The thickening M^c_φ is the restriction of M to the subgraph $(K - \bigcup \alpha) \cup \bigcup T$. The discs of the handle decomposition of M^c_φ are defined as the subthickenings M_T and the strips are defined as the strips of M not contained in the subthickenings M_α and M_T.

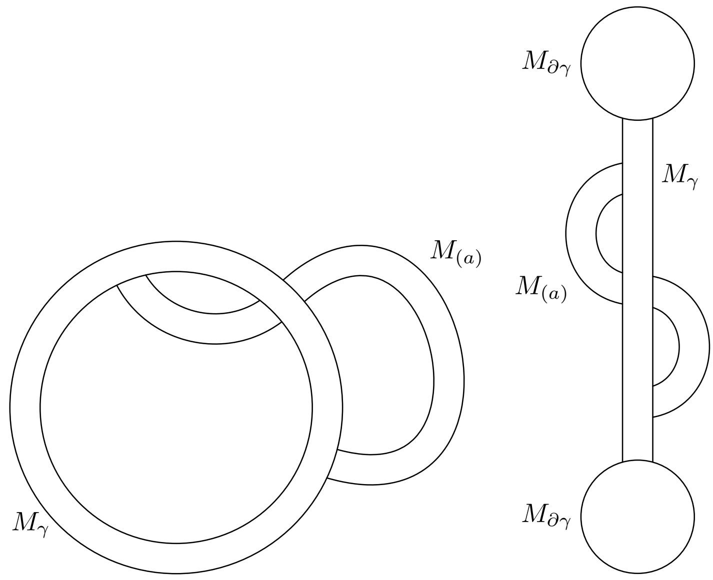

FIGURE 12. The thickenings

Definition (Definition of $\bar{M}'$, cf. Definition of $\bar{\varphi}'$ in §2)**.** Let the map $\varphi : K \to G$ be nondegenerate and locally approximable by embeddings. The thickening $\bar{M}'_\varphi$ and its handle decomposition are defined as follows. Take a disjoint union of M_α for all φ-components $\alpha \subset K$. If φ-components α and β have a common vertex x, then join M_α and M_β by a strip attached to M_α and M_β along the arcs $M_x \cap M_{(a)}$ and $M_x \cap M_{(b)}$ respectively, where $a \subset \alpha$ and $b \subset \beta$ are any edges containing x. The handle decomposition of $\bar{M}'_\varphi$ is obtained from those of M_α by adding the attached strips to the decomposition.

We omit φ from the notation of M^c_φ, $\bar{M}'_\varphi$ and M'_φ. We denote *the derivative* of the thickening M by $M' = ((\bar{M}^c)')^c$. Note that M' is a thickening of a graph which may be different from K'_φ only in multiplicity of some edges. Now we are going to prove the following Lemma 4.5.C,D and Lemma 4.6.C,D, analogous to 2.5.A, 2.1, 2.5.V and 2.2.V respectively.

Lemma 4.5. *Let $\varphi : K \to G$ be locally approximable by embeddings. Then:*

C) φ^c is approximable by embeddings $M^c \to N$ if and only if φ is approximable by embeddings $M \to N$;

D) if $G \cong I$ or $G \cong S^1$ and $\bar{\varphi}'$ is approximable by embeddings $\bar{M}' \to N'$, then $\bar{\varphi}$ is approximable by embeddings $\bar{M}' \to N$.

Proof of 4.5.C. The sufficiency is obvious because $M^c \subset M$ (without handle decompositions). Let us prove the necessity. Let $f : M^c \to N$ be an embedding, S-close to φ^c. Identify M^c and the subthickening $M_{\bar{K}}$, where $\bar{K}$ is the unionwhere $\bar{K} = (K - \bigcup \alpha) \cup \bigcup T$ (see the notation in Definition of M^c). Let us add to $\bar{K}$ the edges a from $\alpha - T$ one by one and extend the map f to the corresponding strips $M_{(a)}$ in an arbitrary way. We assert that an embedding $M \to N$, S-close to φ, can be constructed in this way. Indeed, assume that algorithm does not work, i. e. at some step the obtained S-close to φ embedding $M_{\bar{K}} \to N$ cannot be extended to $M_{(a)}$. Then the two arcs $f\partial M_{(a)}$ are contained in distinct connected components of $\mathrm{Cl}(N_{\varphi\alpha} - fM_{\bar{K}})$. Since the arcs $f\partial M_{(a)}$ belong to one connected component of $N_{\varphi\alpha} \cap fM_{\bar{K}}$, it follows that there are only two possibilities: 1) $\bar{K}$ contains a cycle γ such that $\varphi\gamma = \varphi\alpha$ and $M_{(a)}$ joins the two components of ∂M_γ (Fig. 12.a); 2) $\bar{K}$ contains an arc γ such that $\varphi\alpha \notin \varphi\partial\gamma$ and $M_{(a)}$ joins the two components of $\partial M_\gamma - M_{\partial\gamma}$ (Fig. 12.b). Clearly, in both cases 1) and 2) the map φ is not locally approximable by embeddings. This contradicts the assumption of the lemma, so the algorithm above gives us the required embedding $M \to N$, S-close to φ. □

Proof of Lemma.4.5.D. Note that N' is well-defined because $G \cong S^1$ and hence $\varphi i \cap \varphi j$ is not transveral for any pair of arcs $i, j \subset K$. Take an S-close to $\bar{\varphi}'$ embedding $\bar{M}' \to N'$. Let $f : \bar{M}' \to N$ be the composition of this embedding with the embedding $N' \to N$ (see Definition of N'). It remains to construct a new handle decomposition $\bar{S}$ of N such that f is an $\bar{S}$-approximation of $\bar{\varphi}$.

Denote by $m = \bar{M}'$. For each vertex $x \in G$ let $\bar{N}_x$ be a small neighbourhood in N of the set $fm_{\bar{\varphi}^{-1}x} \cup N_x$. Since $G \cong S^1$, it follows that $\bar{\varphi}^{-1}x$ is a disjoint union of arcs and hence $\bar{N}_x$ is a disjoint union of discs. For each edge $xy \subset G$ the set $fm_{\bar{\varphi}^{-1}xy} - \bar{N}_x$ is also a disjoint union of discs, because for each edge a such that $\bar{\varphi}a = xy$ the strip $fm_{(a)}$ joins $\bar{N}_y$ and some disc of $\bar{N}_x$. Hence $fm_{\bar{\varphi}^{-1}xy} - \bar{N}_x$ does not decompose N, and since $\bar{N}_x$ is a disjoint union of discs, it follows that $\bar{N}_y \cup (fm - \bar{N}_x)$ does not decompose $N_{(xy)}$. So if $\bar{N}_x$ is not connected, one can join any of its connected components $C \subset N_{(xy)}$ with another connected component of $\bar{N}_x$ by a strip in $N_{(xy)}$ not intersecting fm and $\bar{N}_y$. Clearly, $\bar{N}_x$ remains to be a disjoint union of discs after this operation. Let us add to each $\bar{N}_x$ such strips until all $\bar{N}_x$ become connected. Then the obtained discs $\bar{N}_x$ and the strips $\bar{N}_{(xy)} = \mathrm{Cl}(N_{(xy)} - \bar{N}_x - \bar{N}_y)$ form the required handle decomposition $\bar{S}$. □

Lemma 4.6. *If there is a* mod 2*-embedding $M \to N$ S-close to φ, then there is a* mod 2*-embedding, S-close to C) φ^c; D) $\bar{\varphi}'$.*

Proof of 4.6.C. Take an S-close to φ mod 2-embedding $M \to N$. Denote by $f : K \to N$ the restriction of the embedding. Make the move from Contracting Edge Proposition 2.5.K (Fig. 3(b)) for each edge of the tree $T \subset K$ (see Definition of M^c). Let $f : K^c_\varphi \to N$ be the restriction of the constructed map. By the construction the local ordering of the edges of K_φ in M^c and the local ordering of their f-images coincide. So the map $f : K^c_\varphi \to N$ extends to the required mod 2-embedding $M^c \to N$. □

We prove Lemma 4.6.D only in case when K does not contain pairs of arcs i, j with $\varphi i \cap \varphi j$ transversal (this assumption for $G \cong S^1$ is satisfied automatically.) The lemma is proved analogously for an arbitrary graph K, if we use the definition of N' from [6].

Proof of 4.6.D. Take an S-close to φ mod 2-embedding $M \to N$. Denote by $f : K \to N$ the restriction of the embedding. Let $\bar{f}' : \bar{K}'_\varphi \to N'$ be the map from Definition of $\bar{f}'$ in §2. By the construction of Definition of $\bar{f}'$ and Definition of $\bar{M}'$ the local ordering of the edges of $\bar{K}'_\varphi$ in $\bar{M}'$ and the local ordering of their $\bar{f}'$-images coincide. So the map $\bar{f}'$ extends to the required mod 2-embedding $\bar{M}' \to N'$. □

Proof of 1.5. The necessity follows from the necessity of the condition $v(\varphi) = 0$ for approximability by embeddings [8], Example 1.1 [12] and Lemma 2.2.A [6]. Let us prove the sufficiency. Suppose that $v(\varphi) = 0$. By Lemma 4.3 there exist a thickening M of K and a mod 2-embedding $f : M \to N$, S-close to φ. Local Approximation Lemma 4.4 implies that φ is locally approximable by embeddings. Note that the graphs K'_φ and $((\bar{K}^c)')^c$ are the same modulo multiplicity of some edges, φ' and $((\bar{\varphi}^c)')^c$ coincide modulo this difference. Change φ' to this *quasi-derivative* $((\bar{\varphi}^c)')^c$. By Lemma 4.6.C,D it follows that for each natural i the derivative thickening $M^{(i)}$ is well-defined and there exists an S-close to the map $\varphi^{(i)}$ mod 2-embedding, and the map $\varphi^{(i)}$ is locally approximable by embeddings. By Lemma 4.2.T *the derivative* $\varphi^{(k)}$ is either "empty" or the standard d-winding for some $d \neq 0$, hence *the quasi-derivative* $\varphi^{(k+1)}$ is either "empty" or the standard d-winding with the same d. By the assumption of the theorem d is either even or ± 1. Since for the standard winding of an even degree $d \neq 0$ the van Kampen obstruction is nonzero and $\varphi^{(k+1)}$ is approximable by mod 2-embeddings, it follows that d is not even. Hence either $d = \pm 1$ or $\varphi^{(k+1)}$ has an empty domain. In both cases there exists an S-close to the map $\varphi^{(k+1)}$ embedding $K^{(k+1)} \to N^{(k+1)}$, where *the quasi-derivatives* $K^{(i)}$ are defined analogously to $\varphi^{(i)}$. Since $K^{(k+1)} = \emptyset$ or $K^{(k+1)} \cong S^1$, it follows that this embedding extends to an embedding $M^{(k+1)} \to N^{(k+1)}$. Applying Lemma 4.5.C,D $k + 1$ times, we get an embedding $M \to N$, which is S-close to φ. The restriction $K \to N$ of the embedding is S-close to φ, and hence φ is approximable by embeddings. □

For the proof of Lemma 4.2.I we need the following definition.

Definition (Definition of φ^s)**.** For a nondegenerate simplicial map $\varphi : K \to G$ denote by $\tilde{\Delta}(\varphi) = \{\, (x, y) \in \tilde{K} \mid \varphi x = \varphi y \,\}$ the *singular graph*, where $\tilde{K}$ is the deleted product of the graph K. The vertices of $\tilde{\Delta}(\varphi)$ are the pairs (x, y) of vertices of K such that $\varphi x = \varphi y$. For an arbitrary simplicial map $\varphi : K \to G$ define the simplicial *singular* map $\varphi^s : \tilde{\Delta}(\varphi^c) \to G$ by $\varphi^s(x, y) = \varphi^c x$ for each vertex $(x, y) \in \tilde{\Delta}(\varphi^c)$.

Proof of 4.2.I. First note that if φ does not identify triods then φ^s does not contain a simple triod. Secondly note that $(\varphi')^s = (\varphi^s)'$ for any simplicial map φ (formally, it follows from [5, Proposition 2.11] for the pair of maps φ, ψ, where ψ is the projection $\psi : \tilde{K} \to K$). Thirdly note that $|\tilde{\Delta}(\varphi^c)| \le k$ and φ^s is a standard winding for a standard winding φ. Therefore, by Lemma 4.2.T, $(\varphi^{(k)})^s$ is a standard winding, and hence $\varphi^{(k)}\Sigma(\varphi^{(k)})$ is a disjoint union of circles. Now note that $\varphi^{(k)}\big|_{\Sigma(\varphi^{(k)})}$ is ultra-nondegenerate (because $(\varphi^{(k)})^s$ is not ultra-nondegenerate in the opposite case) and $\Sigma(\varphi^{(k)})$ has no hanging vertices (because $\tilde{\Delta}(\varphi)$ has a hanging vertex in the opposite case). So $\varphi^{(k)}\big|_{\Sigma(\varphi^{(k)})}$ is an ultra-nondegenerate map of a disjoint union of circles into a disjoint union of circles $\varphi\Sigma(\varphi)$, consequently $\varphi^{(k)}\big|_{\Sigma(\varphi^{(k)})}$ is a standard winding. □

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