CIS-GRAPHS

Recall some basic definitions. Let G be a graph.

Consider some subset $V = \{v_1, \ldots, v_n\}$ of vertices of G. A *subgraph* of G on the set V is a graph G' having V as the set of vertices, and two vertices of G' are connected if and only if they are connected in G.

A *complement of* G is a graph with the same vertices as G has and with those and only those edges which do not appear in G.

Consider some subset of vertices. If every two vertices from this subset are connected, then such subset is called a *clique*; if there are no edges between these vertices, then this subset is called *stable*. A clique (a stable set) is *maximal* if it is not contained in a larger clique (stable set).

The main definition in this series of problems is a definition of a CIS-graph.

We call G a *CIS-graph* if $C \cap S \neq \varnothing$ for every maximal clique C and every stable set S. E.g., a cycle of length 4 is a CIS-graph, while a Π-graph (see Fig. 9) is not a CIS-graph.

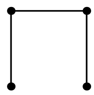


Fig. 1

Is the set of CIS-graphs numerous? Problems needing only the definition.

1. Show that a disjoint union of two CIS-graphs G_1 and G_2 is again a CIS-graph.
Remark. A disjoint union of graphs G_1 and G_2 is a union with no coinciding vertices and no edges between G_1 and G_2.

2. Find all CIS-graphs which do not contain a clique on 3 vertices. Prove that one can colour the vertices of such graph in two colours so that each two connected vertices share different colours.

3. Suppose that one deletes from a CIS-graph a vertex incident to exactly one edge. Prove that the resulting graph in also a CIS-graph.

4. a) Call a vertex *good* if any two its neighbours are connected (i.e., this vertex, together with all its neighbors, forms a clique). Prove that if each maximal clique contains a good vertex, then the graph is a CIS-graph.

b) Show that the converse does not hold.

5. Consider a graph whose set of vertices is a union of clique and a stable set, and these clique and stable set have a common vertex. Show that this graph is a CIS-graph.

We mean by a *substitution* of a graph G_1 into a graph G_2 the following operation: a fixed vertex A of graph G_2 is replaced by a graph G_1, and vertices $B \in G_1$, $C \in G_2 \setminus \{A\}$ are connected if and only if A was connected with C.

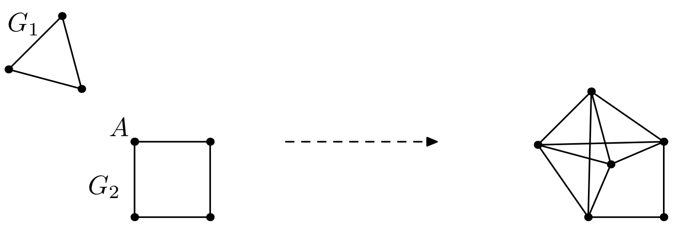


Fig. 2

6. Suppose that the result of substitution of G_1 into G_2 is a CIS-graph. Show that both G_1 and G_2 are CIS-graphs.

7. Show that each graph is contained as a subgraph in a CIS-graph.

8. Consider two sequences of graphs $G_1, G_2, G_3, \ldots$ and $H_1, H_2, H_3, \ldots$ such that either both sequences are infinite, or both consist of equal number of elements. Consider the set of graphs G for which the following condition holds: For each subgraph $G' \subset G$ isomorphic to G_i, there exists a subgraph $H' \supset G'$ isomorphic to H_i.

a) Construct sequences (G_i) and (H_i) in such a way that the resulting set of graphs is exactly the set of all CIS-graphs.

b) For the same purpose, can these sequences of graphs be finite?

Hereinafter, we will say simply "a graph G contains H" instead of "a graph G contains a subgraph isomorphic to H".

9*. Suppose that a graph G contains exactly one pair of non-intersecting maximal clique C and maximal stable set S.

a) Prove that G cannot contain exactly one vertex not belonging to $C \cup S$.

b) The same for two vertices.

c) Try to prove that the set of vertices of G coincides with $C \cup S$.

Combs and settled combs.

A family of graphs is called *closed under substitution* if for any two graphs G_1 and G_2 from this family, the result of arbitrary substituting of G_1 into G_2 also belongs to this family. A family of graphs is called *exactly closed under substitution* if a graph from this family can be obtained by substitution only from graphs of this family.

A family of graphs is called *closed under complementation* if for any graph from this family, its complement also belongs to this family.

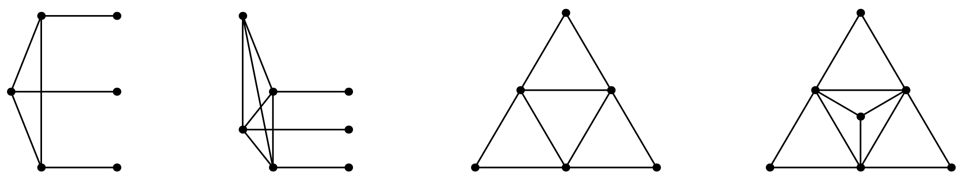
A family of graphs is called *hereditary* if for any graph from this family, all its subgraphs also belong to this family.

It is obvious that the family of CIS-graphs is closed under complementation. The result of problem 6 says that it is closed under substitution. Let us find some other properties of CIS-graphs.

10. a) Suppose that a graph contains a pair of non-intersecting maximal clique and maximal stable set. Prove that this graph contains a Π-graph as a subgraph.

b) Prove that converse does not hold.

Definition. We mean by a *k-comb* ($k \geqslant 2$) the following graph on $2k$ vertices. The first k vertices form a clique, the last k vertices form a stable set, and for each $i = 1, 2, \ldots, k$ the ith vertex from the second set is connected only with the ith vertex from the first set. We mean by a *settled k-comb* the graph obtained from the k-comb by adding a vertex; the new vertex is connected with all the vertices of the clique, and only with them. By a *k-anticomb* and a *settled k-anticomb*, respectively, we mean complements to the k-comb and the settled k-comb.



3-comb   settled 3-comb   3-anticomb   settled 3-anticomb

11. a) In a CIS-graph, any Π-subgraph is contained in an A-subgraph (see Fig. 3).

b) In a CIS-graph, any subgraph isomorphic to a k-comb is contained in a subgraph isomorphic to a settled k-comb.

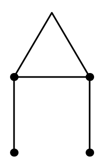


Fig. 3 . A-graph

12. Construct a graph in which for all k, each k-comb is contained in a settled k-comb, each k-anticomb is contained in a settled k-anticomb, and there exists a pair of non-intersecting maximal clique and maximal stable set.

Guiding light. Try to prove the following important theorem.

Theorem. Suppose that a graph does not contain 3-comb and 3-anticomb as subgraphs; moreover, suppose that each 2-comb is contained in a settled 2-comb. Then this graph is a CIS-graph.

Definition. A complete graph with all its edges coloured in d colours will be referred to as a *d-graph*. In a d-graph G, denote by E_i the set of all edges sharing ith colour. A graph having the same set of vertices, while its set of edges is E_i, will be referred to as a *chromatic* (or *colour*) *component* of ith colour.

Definition of a CIS-d-graph. Consider a d-graph. For each colour, choose an arbitrary maximal stable set in ith chromatic component. If for every such choice all these stable sets will share a common vertex, then G is called a *CIS-d-graph*.

Consider a game with complete information for d men. One can represent it as a finite tree with vertices being the positions in this game. There is a root vertex (an initial position of the game); all edges outcoming from this vertex are coloured in the 1st colour — this is the colour of the player making the first move; doing this, the player chooses the next position in the game. For each obtained vertex, there are some edges of the next player's colour (for different new vertices, these colour may differ; thus, the first player by his move chooses the player making the next move). After that the game proceeds in the same way. Since the tree is finite, the game will necessarily end after some moves. The terminal vertex (the result of the game) has no outcoming edges. One can put into correspondence to this tree a d-graph in the following way: The vertices are the terminal vertices of the tree. Consider two vertices and the (unique) path connecting them in a tree; consider the vertex of this path which is closest to the root; suppose that its outcoming edges share ith colour. Then connect our two vertices with an edge of ith colour.

One can show that the resulting d-graph is a CIS-d-graph (you can try to think this over).

Consider a CIS-d-graph. Each its chromatic component is itself a CIS-graph (problem), but it is not known whether the converse holds. While adding the assumption that a CIS-d-graph does not contain a 3-coloured triangle, this converse statement appears to be true (problem). A description of such CIS-d-graphs can be reduced to a description of CIS-graphs (though there is no admissible description to either).

Up to this moment, it is unknown whether there exists a CIS-d-graph with a distinct-colored triangle; it can be checked (involving computer) that there is no such graph with ≤ 12 vertices.

The properties of Gallai graphs.

By a Δ-graph we mean a graph on 3 vertices, with edges sharing 3 distinct colours (a 3-coloured triangle).

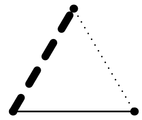


Fig. 4 . Δ-graph

Definition. A d-graph not containing Δ-graph as a subgraph is called a Gallai d-graph.

Let us redefine a Π-graph as a graph on 4 vertices, in which edges of one colour form a Π, while all other edges share the second colour.

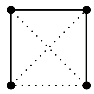


Fig. 5 . Coloured Π-graph

The aim of this section is formulated in the last problem; this problem involves a magnificent property of d-graphs — a Gallai decomposition, which is described in problem 16.

13. Suppose a d-graph G satisfies the following property: if one deletes all the edges of one arbitrary colour, then the graph remains connected. Assume in addition that G is neither a Π-graph nor a Δ-graph. Prove that one can delete a vertex from this graph so that the resulting graph also satisfies this property.

14. Suppose that each chromatic component of a d-graph ($d \geq 3$) is connected. Prove that this graph contains a Δ-graph.

15. Check whether a class of Gallai d-graphs is closed under substitution, exactly closed under substitution, and whether it is hereditary.

16. Prove that every Gallai d-graph can be obtained as a result of a substitution of n d-graphs into some 2-graph (instead of n its vertices).

17. Let F be a class of graphs, which is exactly closed under substitution, and closed under complementation. Consider a Gallai d-graph such that all its chromatic components except at most one (suppose of dth colour) belong to the class F. Suppose that the given graph has at least one edge of dth colour. Then the dth chromatic component also belongs to F.

After semi-final.

Gallai CIS-d-graphs.

18. Check whether the class of CIS-d-graphs is closed under substitution, exactly closed under substitution and hereditary.

19. If a given d-graph does not contain Δ-graphs and Π-graphs as subgraphs, then it is a CIS-d-graph.

20. Show that a d-graph corresponding to a game with complete information (see the first section) does not contain a Δ-graph and a Π-graph as subgraphs, and, conversely, every such graph corresponds to a game.

21. Give an example of Gallai d-graph, but not a CIS-d-graph, such that it has edges of at least 3 different colours.

22. Give an example of CIS-d-graph, which has edges of at least 3 different colours.

23. A Gallai d-graph is a CIS-d-graph if and only if all its chromatic components are CIS-graphs.

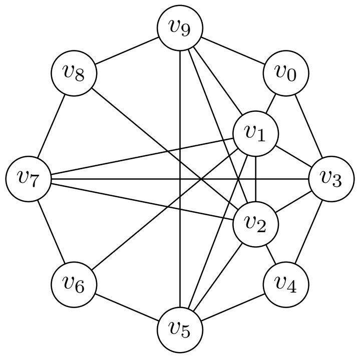
24. If all but one chromatic components of a Gallai d-graph are CIS-graphs then this d-graph is a CIS-d-graph.

25. Assume that all CIS-3-graphs are Gallai 3-graphs. Prove that all CIS-d-graphs are Gallai d-graphs.

26. Hypothesis. Each CIS-d-graph is a Gallai d-graph.

Per aspera ad astrum.

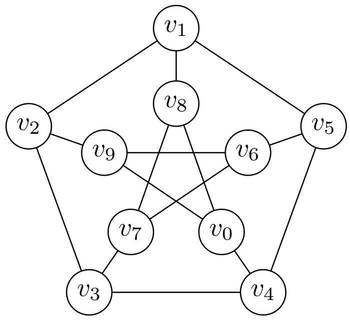
Definition. Let G_{10} be a graph drawn in the picture below:



G_{10}

Let $2G_{10}$ be a graph with twenty vertices $v_0, v_1, \ldots, v_9, v'_0, v'_1, \ldots, v'_9$, which is obtained from two copies of G_{10} (the first graph G_{10} on vertices $v_0, v_1, \ldots, v_9$, the second on the remaining vertices). For $i \neq j$ we draw an edge from v_i to v'_j if and only if v_i and v_j are not connected with an edge. Edges between v_i and v'_i may exist or may not exist (independently for each i). So, $2G_{10}$ means an arbitrary one of these 1024 graphs.

The graph in the figure below is the Petersen graph. We denote it by P. Take two its copies on the sets of vertices $u_1, u_2, \ldots, u_{10}$ and $v_1, v_2, \ldots, v_{10}$, respectively. For $i \neq j$, we link u_i and v_j with an edge if u_i and u_j are not linked with an edge. For any pair of vertices u_i and v_i, we may either link them or not. If we choose one of these $2^{10} = 1024$ graphs, then the obtained graph is named "$2P$".



The Petersen graph

Theorem. Suppose that a graph does not contain 3-comb and 3-anticomb as subgraphs; moreover, suppose that each 2-comb is contained in a settled 2-comb. Then this graph is a CIS-graph.

The remaining problems are devoted to the proof of this theorem.

27. Looking for G_{10}. Suppose that G has at least one pair of non-intersecting maximal clique C and maximal stable set S, G does not contain 3-comb and 3-anticomb as subgraphs, and each 2-comb is contained in a settled 2-comb.

a) Show that there exist vertices in G belonging neither to C nor to S.

b) Show that G contains G_{10} as a subgraph.

28. Find the second!

Suppose that in G all 2-combs are contained in settled 2-combs, G contains G_{10} on vertices $v_0, v_1, \ldots, v_9$ and does not contain 3-combs and 3-anticombs as subgraphs.

a) Show that G contains ten vertices $v'_0, v'_1, \ldots, v'_9$ such that for every distinct i and j vertices v_i and v'_j are linked if and only if vertices v_i and v_j are not linked.

b) Find a subgraph $2G_{10}$ in G.

Where are the combs? Notice that a graph $2G_{10}$ does not contain a 3-comb and a 3-anticomb, but it contains 2-combs, not included in settled 2-combs, if we suppose, for example, that v_2 and v'_2 are linked with an edge.

In a problem below, all 2-combs in G are contained in settled 2-combs, it contains $2G_{10}$ and does not contain a 3-comb and a 3-anticomb.

29. a) Show that the vertices v_2 и v'_2 of the subgraph $2G_{10}$ are not linked.

b) Show that the vertices of $2G_{10}$ might be renumbered in such a way that we obtain $2P$.

c) Prove the Theorem.

CIS-graphs. Solutions of the problems

We sometimes use a term "anticlique"instead of the words "stable set"in these solutions. A set of all vertices in a graph G is denoted by $V(G)$.

Solution of Problem 1. Let $G = G_1 \sqcup G_2$ be a disjoint union of G_1 and G_2, and let C and S be a maximal clique and a maximal stable set (respectively). It is obvious that S intersects with each of the two graphs G_1 and G_2 and C is contained in one of them. Without loss of generality, let us assume that C is contained in G_1. Then $S \cap G_1$ and $C \cap G_2$ intersect as every pair of a maximal stable set and a maximal clique in G_1.

Solution of Problem 2. If there are no triangles in a CIS-graph, then there are also no Π-subgraphs. Otherwise the two vertices "in the middle"of this subgraph (i.e. whose degrees are 2) form a maximal clique (let us designate it by C). And the two vertices in this subgraph whose degrees are 1 form a stable set which does not intersect with C and is contained in a certain maximal stable set that also does not intersect with C.

Consider every connected component in our graph. If there exist two vertices the distance between which is more than two, then there are two vertices the distance between which equals three. The shortest path connecting them consists of 4 vertices, and a subgraph on these vertices is the Π-subgraph, and this is impossible. So every two vertices in every component of connectedness are connected by a path of length not more than two.

Suppose there exist a cycle of odd length (we call such cycles "odd cycles"), then let us consider the minimal among them. It contains at least 5 vertices, and the pairs of non-adjacent vertices are not connected by edges (otherwise the cycle is divided into two smaller ones, one of them is odd). Thus every four consequent vertices of this cycle form a Π-graph, and this is impossible. So there are no odd cycles. It is not difficult to prove that such graph is bipartite: its vertices may be divided into two parts so that the ends of every edge lie in different parts.

Therefore, every connected component is a bipartite graph. It must be complete (every two vertices from different parts are connected): the length of the minimal path between the two vertices from different parts that are not connected by an edge is not less than 3. So our initial graph is a disjoint union of several complete bipartite graphs. It is very easy to draw the vertices of such graph in two colors to satisfy required condition: in every connected component we draw one part in the first color and the other part in the second color.

To finish the solution it is only necessary to check that every such graph satisfies the conditions of the task. Note that since a complete graph is a CIS-graph then a complete bipartite graph (as a complement to a disjoint union of two complete graphs) is also a CIS-graph. So every disjoint union of complete bipartite graphs is a CIS-graph.

We shall note the following trivial fact: a graph is a CIS-graph if and only if any clique and anticlique can be extended to intersecting maximal clique and anticlique.

Solution of Problem 3. Let us denote a graph which is obtained from a graph G by removing the vertex v from G (and all the edges containing v) by $G-v$. Suppose that G is a CIS-graph, $G-v$ is not and the vertex v is connected to a unique vertex u in the graph G. Then there exist a maximal clique C and a maximal stable set S in $G - v$ that do not intersect. Let us extend them to a maximal clique C' and a maximal anticlique S' in the graph G respectively.

It is clear that C' and S' intersect only by vertex v. But then C cannot contain any vertices except u (because there is only one edge containing v - the one between v and u). Also if a maximal clique consists of a unique vertex then this vertex is isolated (i.e. is not connected to other vertices) and therefore is contained in every maximal anticlique. So C and S intersect by the vertex u. Contradiction.

Solution of Problem 4. a) Suppose that it is not true, then in a graph G there are a maximal clique C and a maximal stable set S which do not intersect, and in C there is a good vertex v. If there is a vertex $u \in V(S)$ connected to v then all the vertices in C are adjacent to v (since v is good), which contradicts the maximality of C. So v is not connected to any of the vertices in S which contradicts the maximality of S.

b) The cycle of length 4 serves as an example.

Solution of Problem 5. Let the maximal clique be C and the maximal stable set be S; they intersect at a vertex u. Then suppose that a certain maximal clique C' and a certain maximal stable set S' do not intersect. If $C' = C$ and S' does not intersect with C' then $S' \subset S$ and therefore $S' = S$. Thus $C' \neq C$ and (due to analogous reasons) $S' \neq S$. Assume that $s \in C' \setminus C,\ c \in S' \setminus S$; we may suppose (without loss of generality) that the edge between c and s does not exist. But since S' is maximal, there must exist a vertex c' adjacent to s in S'. Note that c' cannot lie in S (there already lies c' and S is an anticlique). Therefore c' lies in C. So the clique C and the anticlique S' have vertices s and c' in their intersection, which is impossible (if it were true, the edge (s', c) would exist and not exist simultaneously).

Solution of Problem 6. Denote the obtained graph by the letter G.

Suppose that G_1 is not a CIS-graph, then there exist a maximal clique C and a maximal stable set S in G_1 that do not intersect. Then consider any maximal clique containing C and any maximal anticlique containing S. Their common vertex is connected to the vertices from C and not connected to any of the vertex from S, which contradicts the definition of substitution.

To prove that G_2 must be a CIS-graph it is sufficient to note that during an operation which is inverse to the operation of substitution cliques and anticliques transform into cliques and anticliques, and, what is more, all cliques and anticliques in G_2 can be obtained in such way.

Solution of Problem 7. Consider an arbitrary graph G and for every its maximal clique add a vertex to the graph and connect it to all of the vertices in this clique and only to them (this procedure has to be done once). In the constructed graph (according to the construction) every maximal clique contains only one added vertex, and obviously it is good. Thus (according to the task 4) the constructed graph is a CIS-graph, and the initial graph is its subgraph.

Solution of Problem 8. a) Consider a clique C and an anticlique S of a graph G. Suppose that $V(G) = C \sqcup S$ and every vertex from C is adjacent to at least one vertex from S, and every vertex from S is not adjacent to at least one vertex from C (i.e. C and S are maximal in G). Every graph G can be extended to a graph $H(G)$ by adding a vertex adjacent to all of the vertices from a clique C and only to them. By enumerating all possible arrays $(G, C, S, H(G))$, we get four sequences $\{G_i\}$, $\{C_i\}$, $\{S_i\}$, $\{H_i\}$.

Note that a graph is not a CIS-graph if and only if (according to the note after the solution of problem 2) there are such clique and anticlique in this graph that it cannot be extended in respect to this clique and anticlique in the way described in the previous passage. In other words, a graph is not a CIS-graph if and only if for a certain i it contains a subgraph isomorphic to G_i which cannot be extended to a subgraph isomorphic to H_i. Therefore, the sequences G_i, H_i satisfy the conditions of the problem.

b) Assume that there exist such two finite sequences $\{G_i\}$, $\{H_i\}$ that satisfy the conditions of our problem. Suppose that the maximal number of vertices in every graph G_i, H_i is equal to n. Consider the n-comb and the settled n-comb. It is not difficult to check that if in the settled n-comb every subgraph isomorphic to G_i can be extended to a subgraph isomorphic to H_i then the same holds true for the n-comb. But the settled n-comb is a CIS-graph while the n-comb is not a CIS-graph. We have obtained the contradiction.

Solution of Problem 9. a) Let $C \sqcup S \cup v$ be the set of vertices of G where C is maximal clique and S is maximal anticlique. Denote by C' a subset of vertices of C which are adjacent to v and denote by S' a subset of vertices of S which are not adjacent to v. Then clique $C' \cup v$ must be extended to maximal clique which intersects with S. Then there is a vertex $s \in S'$ which is adjacent to all vertices from C' and to vertex v. Similarly there is a vertex $c \in C'$, which is not adjacent to all vertices from S' and to vertex v. Without loss of generality vertices s and c are linked. So there is a vertex $c' \in C$, which is not adjacent to s (so it follows that c' is not adjacent to v). Consider clique C'' on vertex s and on all its neighbors in C and any maximal anticlique S'' that contains vertices v and c'. Vertex v can not be added to clique C'' because $c \in C''$. So this clique is maximal. Any vertex of C'' is adjacent to v or to c'. Then S'' does not intersect C''.

b),c) Consult the article cis1.pdf.

Solution of Problem 10. a) (*The solution was proposed by the following team: Vasiliy Mokin, Viktor Omelyanenko and Viktor Sadkov.*) We denote this pair of sets by C and S and consider a complete oriented bipartite graph with a set of vertices $C \cup S$ and edges drawn according to the rule: if the vertices $c \in C$ and $s \in S$ are adjacent then the edge in the new graph is directed from c to s, otherwise it is directed from s to c. Since S is a maximal anticlique then every vertex in the initial graph is adjacent to at least one vertex from S. In particular, this means that in the new graph for every vertex in C there exists an edge that is directed from this vertex. Analogously, for every vertex in S there exists an edge that is directed from this vertex.

Thus in our oriented graph for every vertex the number of edges directed from it is not less than 1, therefore, there is a cycle in our graph (by a cycle in an oriented graph we mean the oriented cycle). Then consider a cycle of minimal length $A_1A_2 \ldots A_{2n}$ (the length is even since the graph is bipartite). If $2n > 4$ then depending on orientation of the edge A_1A_4 it is possible to find a cycle smaller than minimal in our graph: either $A_1A_2A_3A_4$ or $A_1A_4A_5 \ldots A_{2n}$. So $2n = 4$. Then it is not difficult to check that the Π-graph was induced on the vertices A_1, A_2, A_3, A_4 in the initial graph.

b) For instance, the A-graph, in other words, the settled 2-comb.

Solution of Problem 11. a) It is a partial case of the next subproblem.

b) Consider an arbitrary maximal clique containing the first group of vertices in the comb (they form a clique) and any maximal anticlique containing the second group of vertices in the comb (they form an anticlique). Their intersection contains at least one vertex, which forms a settled comb by being added to the comb.

Solution of Problem 12.

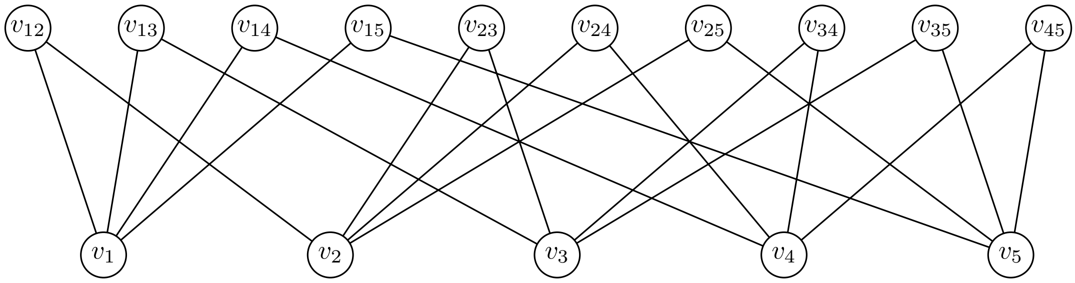


Fig. 6

The main property of Gallai graphs.

Solution of Problem 13. Denote the given d-graph by G, and let v be one of its vertices. Notice that there exists at most one colour such that if we exclude v from G then the subgraph of not this colour is not connected: If G becomes non-connected when we omit the edges of the 1st colour, then all the edges between its connected components are of the 1st colour. But the full bipartite graph is connected, hence if we omit some other colour then the remaining graph will be connected. Suppose that for any vertex v there exists a colour such that $G - v$ is not connected without this colour. We showed that for each vertex there exists at most one such colour (if for a vertex there are no such colours, then we are done). Colour this vertex in this colour.

Notice that in any connected graph there exist at least two vertices such that deleting of them does not break connectedness. To show this, take the spanning tree of this graph. Its terminal vertices can be deleted. In our case, the initial graph without any fixed colour is connected, this means that for this colour there exist at least two vertices such that deleting of them does not break the connectedness of the graph, i.e. coloured not in this colour.

Suppose $u$is a vertex of the 1st colour. Let $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_m$ be a splitting of all the other vertices of the graph such that these components are not connected without the 1st colour (in $G - u$). All the edges between these two groups are of the 1st colour. We will name these groups by *parts*; a part containing at least two vertices will be named *big*. If both parts are small, then either the given d-graph is a 3-coloured triangle or it does not satisfy the conditions of the problem. Suppose that there exists at least one big part. We have at least two vertices not of the 1st colour, while a small part can contain at most one of these vertices. This means that at least one big part contains a vertex not of the 1st colour. Without loss of generality, suppose it is the second part, i.e. $m \geqslant 2$, and $w_1$is of the 2nd colour. Perform the same procedure for the 2nd colour. The vertices of G split into two parts, such that all the edges between these parts are of the 2nd colour. But the edges $(w_2, v_1), (w_2, v_2), \dots, (w_2, v_n), (v_1, w_3), (v_1, w_4), \dots, (v_1, w_m)$ are of the 1st colour, this means that $w_2, v_1, v_2, \dots, v_n, w_3, \dots, w_m$ are in one part with respect to w_1. Hence, u is in the other part, and all the edges from u to all the other vertices, excluding w_1, are of the 2nd colour.

Consider the 2nd vertex w_1' of colour $k \neq 1$. Perform the same procedure as with w_1: Split all the other vertices into two parts and look in which part with respect to u it belongs. If it belongs to a big part, then, similarly, u is connected with all vertices but w_1' with the edges of colour k. Notice that this graph contains some other vertices besides u, w_1, and w_1' (since G contains at least 4 vertices). They are connected with u with the edges of colours 2 and k simultaneously, hence, $k = 2$. It follows that all the edges incident to u are of the 2nd colour. Hence, the initial d-graph is not connected if we exclude the edges of the 2nd colour. Contradiction. We obtain that $w_1' = v_1$, $n = 1$, and the vertices $w_2, w_3, \dots, w_m$ are all of the 1st colour.

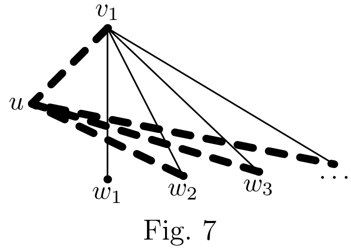


Fig. 7

Delete w_2. The vertices of the graph split into two parts in such a way that all the edges between them are of the 1st colour. Since all the edges $(u, v_1), (u, w_2), (u, w_3), \dots, (u, w_m)$ are of the 2nd colour, all the vertices $u, v_1, w_3, \dots, w_m$ are in one part. Hence, w_1 is in the other part. We obtain that w_1 is connected with the vertices $u, v_1, w_3, w_4, \dots, w_m$ with the edges of the 1st colour. If $m > 2$, we obtain in the same way that the edge (w_1, w_2) is of the 1st colour, which means that all the edges incident to w_1 are of the 1st colour. Contradiction. Then $m = 2$. If the edge (w_1, w_2) is of the 2nd colour, then the given d-graph is a Π-graph. If the edge (w_1, w_2) is of the 1st colour, then the initial d-graph without the edges of the 1st colour is not connected. This is also a contradiction.

Solution of Problem 14. Induction on the number of vertices in the graph. The base for $n = 3$ is obvious: at least one chromatic component is not connected. Perform the step. Suppose that $n > 3$ and all chromatic components are connected. Exclude one vertex (let a); if all the chromatic components remain connected, then we use the statement of induction and find the 3-coloured triangle. Otherwise one of the chromatic components (suppose the 1st) became disconnect. Let $C_1, \dots, C_k$ be the connected components of the 1st colour in the obtained graph; notice that for each of these components there exists an edge from a to this component of the 1st colour.

Consider two components C_i and C_j and an edge connecting two their vertices c_i and c_j; without loss of generality, let it be of 2nd colour. Take another vertex c_i' from C_i. If it is connected with c_j with an edge of the 3rd colour, then we get a 3-coloured triangle. It means that the edge $c_i' - c_j$ is also of 2nd colour, and so on. One gets that for any two components C_i and C_j all the edges between them are of the same colour.

Now return to the initial graph. All its chromatic components are connected, this means that a has an edge of colour 2. Let this edge join a with a vertex $c_r \in C_r$. If there exists a component C_s such that the edges between C_s and C_r are of another colour (denote this colour 3), then find a vertex $c_s \in C_s$ such that the edge $a - c_s$ is of colour 1 (this is possible because the 1st connected component of the initial graph is connected); then a, c_r, c_s form a 3-coloured triangle.

We showed that all the edges joining C_r with all the other components are of colour 2. Similarly, if we take an edge of colour 3 incident to a and consider the component C_t incident to the second end of this edge, then we obtain that all the edges connecting C_t with other components are of colour 3. But this is impossible: if $r = t$, notice that we have more than one component, and it is not clear edges of which colour link this component with C_r. If, otherwise, $r \neq t$, then it is not clear edges of which colour join C_r and C_t.

Solution of Problem 15. As in the solution of problem 6, we show that this family is closed under substitution. To show the exact closeness, notice that for any vertex from the d-graph we can find d sets each of which will be independent of the edges of its colour and will be maximal with this property. When $d = 2$, as we noticed above, this family is not hereditary.

Solution of Problem 16. An obvious Lemma. Suppose that a Gallai d-graph has an edge of dth colour and its dth chromatic component is not connected. Then all the edges between any two fixed connected components of this chromatic component are of the same colour.

Its proof is a part of proof of 14.

The solution of the problem. We prove, using descent on k ($k < d$) the following statement:

Suppose we have a Gallai d-graph. Then it is the result of substitution of d-graphs $G_1, G_2, \dots, G_n$ instead of n vertices of a k-graph G, and G has at least two vertices.

This statement is obviously true for $k = d$. Suppose it is true for a given k, prove it for $k-1$. Take a Gallai d-graph and construct for this graph and for k the decomposition described above. From the previous problem it follows that the obtained k-graph G and all the substituted subgraphs $G_1, G_2, \dots, G_n$ are Gallai d-graphs. From problem 14 it follows that one of the chromatic components of the k-graph G is not connected. Let it be the kth component and $C_1, C_2, \dots, C_m$ be the connected components of this chromatic component. We denote by F_i the subgraph of G with the set of vertices equal to C_i. Using the Lemma, we obtain that G is the result of substitution of some graphs $F_1, F_2, \dots, F_m$ instead of m vertices of a $(k-1)$-graph H. If we finally substitute the graphs $G_1, G_2, \dots, G_n$ instead of some vertices of graphs $F_1, F_2, \dots, F_m$ (if C_i consists of vertices numbered $i_1, i_2, \dots, i_p$, then we should substitute $G_{i_1}, G_{i_2}, \dots, G_{i_p}$ in F_i according to the numbers of vertices). Suppose that we obtained graphs $H_1, H_2, \dots, H_m$. Then the initial Gallai d-graph is the result of substitution of d-graphs $H_1, H_2, \dots, H_m$ instead of vertices of a $(k-1)$-graph H. We prove the necessary statement when $k = 2$.

Solution of Problem 17.

Лемма. *Suppose that a given Gallai graph has edges of colour d and $d \geqslant 3$. Then a graph on n vertices without edges belongs to F.*

Доказательство. If F contains at least one non-connected graph, then the graph having two vertices and no edges also belongs to F since this graph is the result of substitution of two graphs in a graph having two vertices and no edges, where we take the first component as the first graph and all the other components as the second graph. After this, we can obtain the graph with n vertices and no edges by substituting the graph above into itself.

It remains to show that the family F contains at least one non-connected graph. We denote by G the graph given in the formulation of the problem. Use induction on the number of vertices of G. As we proved in problem 16, G can be obtained as the result of a substitution of some d-graphs $G_1, G_2, \dots, G_n$ instead of n vertices of a 2-graph H with at least 2 vertices. This generates substitutions on the chromatic components of G. Either one of G_is or H has edges of dth colour. Since the family F is exactly closed, all the conditions above remain true for the G_i (or H) found above. If we apply the step of induction, we obtain the required. Now prove the base of the induction. If we take a graph with 2 vertices which has an edge of colour d, then its 1st chromatic component is not connected. □

Now solve the problem using induction on the number of vertices of G.

The base. A graph with one vertex always belongs to a family of graphs exactly closed under substitution.

The step. The first case is if G contains edges of at most two colours. By the conditions, all its chromatic components but at most one belong to F. We have at least one edge of dth colour, this means that the dth chromatic components is a complement to one of the previous chromatic components, hence, it also belongs to F. Now consider the second case, when G has edges of at least 3 different colours. As we showed in the previous problem, we can obtain G as the result of substitution of some d-graphs $G_1, G_2, \dots, G_n$ instead of n vertices of a 2-graph H which has at least 2 vertices. Notice that the chromatic component of G of the ith colour is the result of substitution the chromatic components of the ith colour of d-graphs $G_1, \dots, G_n$ instead of vertices of the ith chromatic component of the d-graph H (any 2-graph is a d-graph which has edges of only two colours). By the condition, all but one chromatic components of G belong to the family F. Since F is exactly closed under substitution, all the chromatic components, excluding, maybe, of colour d, of d-graphs $H, G_1, G_2, \dots, G_n$ belong to F. Each of these graphs has less vertices than G. Using the induction statement and lemma 17, we obtain that the dth chromatic component of each of these graphs belongs to F. Since F is closed under substitution, the dth coloured component of the d-graph G also belongs to the family F.

Solution of Problem 18. Similarly to the solution of Problem 6.

Solution of Problem 19. Follows from Problems 10 and 23.

Solution of Problem 20. If x and y are terminal vertices, denote by $P(x, y)$ the nearest vertex to the root on the path connecting these vertices (it is also denoted their least common parent). It is easy to check that for any two results of the game (which correspond to the terminal vertices of the tree) x, y, z the set of vertices $P(x, y)$, $P(y, z)$, $P(z, x)$ has at most two vertices. Hence, at least two of the edges (x, y), (y, z), (z, x) are of the same colour. In the same way (it can be checked by considering several cases) it can be checked that we cannot find four vertices and colour their parents in such a way that this subgraph on 4 vertices is a Π-graph. **The converse statement.** We use induction on the number of vertices. The base is obvious. The step: we showed in Problem 13 that there exists a colour (suppose the 1st), such that deleting of all the edges of the 1st colour makes the graph non-connected (it is clear that all its components remain free from Δ and Π). Using the statement of the induction, we may construct a tree for each of the connected components. Then we can take the disjoint union of these graphs and add one vertex connected with all the roots of these trees with the edges of the 1st colour. It is clear that we obtained the required tree.

Solution of Problem 21.

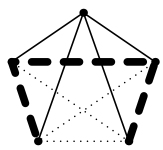


Fig. 8

Solution of Problem 22.

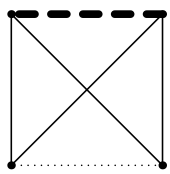


Fig. 9

Solution of Problem 23. Suppose we have a CIS-d-graph. Due to lemma in solution of problem 25 we have all chromatic components are CIS-graphs.

Let all chromatic components be CIS-graphs. Let us prove that our d-graph is a CIS-d-graph. Proof is similar to proof of problem 17. We will make a sketch. The statement is obvious for 2-graphs. If we have edges of at least three colours then we can use problem‘s 16 statement. So let our d-graph is a result of substition of $G_1, G_2, \dots, G_n$ to H, where the numbers of vertices of these graphs are less then the number of vertices of our d-graph. Due to sixth problem all chromatic components of these new graphs are CIS-graphs. So we use induction and obtain that they are CIS-d-graphs. Finally due to problem 18 we obtain that our d-graph is a CIS-d-graph.

Solution of Problem 24. Follows from Problems 23, 17, 6.

Solution of Problem 25.

Лемма. *If we merge two colours of a CIS-d-graph, then it remains a CIS-d-graph.*

Доказательство. Consider a maximal set A of the vertices of this d-graph free from edges of colours 1 and 2. Let B and C be the maximal sets of vertices, such that $B \supseteq A$, $C \supseteq A$, and B and C are free from edges of colours 1 and 2, respectively. Notice that $B \cap C = A$, because in the other case A could be extended. Let $A_3, A_4, \dots, A_d$ be the maximal sets of vertices free from edges of colours $3, 4, \dots, d$ respectively. Since the given graph is a CIS-d-graph, intersection of all the sets $B, C, A_3, A_4, \dots, A_d$ is one vertex, this means that the intersection of the sets $A, A_3, A_4, \dots, A_d$ is also one vertex. Hence, when we merge the 1st and the 2nd colour, the graph remains a CIS-d-graph. □

Show that our graph does not contain 3-coloured triangles with the edges of colours 1, 2, and 3. Merge all the edges of the 4th, $5th, \dots$, and dth colour with the 3rd colour. According to the Lemma, we obtain a CIS-3-graph. By the condition, the CIS-3-graph does not contain 3-coloured triangles. Hence the initial graph also does not contain 3-coloured triangles.

Solution of Problem 27. The solution is rather hard, it's better to consult the article `17_2006.pdf`.

Solution of Problem 28. Consider a vertex v. Denote by $u_1, u_2, u_3, w_1, w_2, w_3$: a+ b- c+, where a,b and c are integers, the following statement: if vertex v is linked with vertices u_a and u_c and it is not linked with vertex u_b then vertices $u_1, u_2, u_3, w_1, w_2, w_3$ form 3-comb or 3-anticomb. More precisely, vertices u_1, u_2, u_3 form a clique, vertices w_1, w_2, w_3 form an anticlique, if vertices $u_1, u_2, u_3, w_1, w_2, w_3$ form 3-comb then vertices v_i and w_j are linked if and only if $i = j$, if they form 3-anticomb then vertices v_i and w_j are linked if and only if $i \neq j$. If $u_1 = v_x$ then denote u_1 by x. If other cases a+ means that vertices v and v_a are linked, and a- — vice versa.

a) Note that G_{10} contains Π-subgraphs which are not contained in A-subgraphs. For every such Π-subgraph we add a vertex which settles Π-graph to A-graph. All but one edges (or non-edges) between added vertex and verteces of G_{10} we will draw because of our graph must not contain 3-comb neither 3-anticomb.

v'_7) Π-subgraph 3 4 5 6 can not be settled to A-subgraph of G_{10}, therefore there exists a vertex v with edges 3- 4+ 5+ 6-.

4 5 v 3 6 8: 8+ Then vertices v and v_8 are not linked, in other words 8-.

Let v and v_1 be linked, in other words 1+.

5 1 v 0 4 6: 1+ 0+ Then 0-.

9 1 v 8 6 4: 1+ 9+ Then 9-.

2 3 4 8 0 v: 2- 0- Then 2+

1 2 9 8 0 v: 1+ 2+ 9- 0- Contradiction. Therefore 1-.

1 5 9 3 v 8: 9-

1 5 9 v 0 6: 9+ 0-

2 9 v 0 4 8: 2+ 9+ 0+

This means that 9+ 0+ 2-. In other words, vertices v and v_i are linked if and only if vertices v_7 and v_i are not linked for $i \neq 7$.

v'_6)Π-subgraph 7 8 9 5 can not be settled to A-subgraph of G_{10}, therefore there exists a vertex v with edges 7- 8+ 9+ 5-.

2 8 9 v 5 7: 2-. In other words 2+.

Let 1+.

1 2 v 8 0 5: 1+ 0+ Then 0-

8 9 v 7 0 4: 0- 4+ Then 4-

1 9 v 8 3 5: 1+ 3+ Then 3-

1 2 3 4 0 v: 1+ 3- 4- 0- Contradiction. Therefore 1-.

1 5 9 7 4 v: 4-

8 9 v 7 0 4: 4+ 0-

1 9 0 v 3 5: 0+ 3-

Consequently we obtain 4+ 0+ 3+. Hence vertices v and v_i are linked if and only if vertices v_6 and v_i are not linked for $i \neq 6$.

Edge (v'_6, v'_7) If not then 3 4 6′ 1 7′ 8

v'_1If we renumber vertices of G_{10}

$$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9) \to (0, 1, 2, 9, 8, 7, 6, 5, 4, 3),$$

we obtain vertices 5′ and 6′, which are linked. Π-subgraph 7 5′ 6′ 9 is not contained on A-subgraph of $G_{10} \cup v'_6 \cup v'_7$, therefore we obtain a vertex v with 7- 5′+ 6′+ 9-.

5 6 8 7 9 v: 8-. Then 8+.

Let 2+.

2 8 v 5′ 4 9: 2+ 4+ Then 4-

3 5 6 v 7 4: 4- 3- Then 3+

2 3 v 0 7 4: 2+ 4- 3+ 0+ Then 0-

6′ 8 v 0 7 5: 0- 5+ Then 5-

2 5 9 0 6′ v: 2+ 5- 0- Contradiction. Therefore 2-.

2 3 6′ v 9 7: 3+ Then 3-

3 5 6 v 7 4: 3- 4- Then 4+

5′ 0 v 7 9 4: 4+ 0+ Then 0-

5′ 8 v 3 9 v: 3- 6+ Then 6-

Vertex v is linked with vertex v_i if and only if vertex v_1 is not linked with v_i.

Other vertices. We can renumber vertices of graph G_{10} in such way

$$(0,1,2,3,4,5,6,7,8,9) \to (4,2,1,5,6,7,8,9,0,3) \to (4,2,1,3,0,9,8,7,6,5,4).$$

So if we obtain vertex $6'$, we automatically obtain vertices $0'$, $8'$, $4'$. Also we already have obtained vertices $7'$ and $1'$, so we obtain all vertices $0', 1', \dots, 9'$.

b) In following items the phrase "If not then"is missed.

Edge (v_7', v_9') 4 $7'$ $9'$ 2 0 6

Edge $(v_6', v_0'))$ $0'$ $6'$ 8 9 7 4

Edge $(v_6', v_8'))$ 1 3 $8'$ $6'$ 5 7

Edge $(v_6', v_9'))$ 4 $6'$ $9'$ 4 0 7

Edge $(v_5', v_9'))$ 3 4 $6'$ $7'$ $5'$ $9'$ because we already have edges $(v_6', v_7'), (v_7', v_9'), (v_6', v_8')$

So we find all edges between vertices $3'$,$4'$,...,$9'$,$0'$.

Edge (2′,4′) 1 0 $4'$ 5 $2'$ 8

Edge (2′,6′) 3 4 $6'$ 7 5 $2'$

Edge (2′,3′) $3'$ 5 6 8 4 $2'$

Using obtained results and renumbering we obtain an edge ($1'$,$2'$): 6 7 $0'$ $2'$ 3 $1'$

This is all.

Solution of Problem 29. a) Let v_2 and v_2' be linked. Consider Π-graph 4 2 2' 0. It can not be settled by a vertex of $2G_{10}$. Then there exists a vertex v such that 4- 2+ $2'$+ 0-.

1) We will prove first that 1-. Assume contrary and we will prove that vertices v and v_i are linked if and only if vertices v_7 and v_i are linked.

So. 1+ 2+ 4- 0-.

1 2 3 4 0 v: 3-. Then 3+.

2 3 v 8 0 6: 6+ 8-

1 3 v 6 4 8: 6- 8+

Then either 6- 8- or 6+ 8+.

Let 6-. Then 8-.

1 2 5 4 6 v: 5- 6- Then 5+

1 2 9 8 0 v: 8- 9- Then 9+

5 9 v 6 8 3: 6- 8- 5+ 9+ Contradiction. Therefore 6+ Й 8+.

2 5 v 6 8 4: 5+ Then 5-.

1 9 v 8 3 5: 9+ Then 9-.

So vertices v and v_i are linked if and only if vertices v_7 and v_i are linked. Consider a new G_{10} which is obtained from the old G_{10} by replacing the vertex v_7 to v.

Vertex 5' is linked with vertices 3 and 0 and it is not linked with vertices 4 and 9. Therefore similarly to finding v_7' in item a) in last problem we obtain that vertex 5' is not linked to vertex v (vertex v takes place of vertex v_7). Vertex 4' is linked with vertices 9 and 0 and it is not linked with vertices 3 and 5. Therefore similarly to finding v_6' in item a) in last problem we obtain that vertex 4' is not linked to vertex v (vertex v takes place of vertex v_7). Vertex 2' is linked with vertices 4' and 5' (vertices 4' and 5' are opposite to vertices 4 and 5 in new G_{10}) and it is not linked with vertices 3 and 9. Therefore similarly to finding v_1' in item a) in last problem we obtain that vertex 2' is not linked to vertex v (vertex v takes place of vertex v_7). Contradiction.

2) Therefore 1-.

Let 6+.

1 5 6 0 4 v: 6+ 5- Then 5+

2 9 v 4 0 6: 6+ 9+ Then 9-

2 5 v 6 8 4: 5+ 6+ 8+ Then 8-

2 5 9 3 6 v: 3+ 5+ 6+ 9- Then 3-

1 5 9 3 v 8: 3- 8- 9- 5+ Contradiction. Then 6-.

2 5 9 v 6 0: 5- 9-

1 5 9 v 0 6: 5+ 9+

Therefore either 5+ 9- or 5- 9+.

1 3 7 v 6 0: 3+ 7+

2 3 7 v 0 6: 3- 7-

Therefore either 3+ 7- or 3- 7+.

1 3 7 9 4 v: 3- 7+ 9-

1 5 9 7 4 v: 5- 7- 9+

Therefore either 3+ 5+ 7- 9- or 3- 5- 7+ 9+.

3) Let 3- 5- 7+ 9+. So vertices v and v_i are linked if and only if vertices v_8 and v_i are linked. Consider a new G_{10} which is obtained from the old G_{10} by replacing the vertex v_8 to v.

Vertex 5' is linked with vertices 3 and 0 and it is not linked with vertices 4 and 9. Therefore similarly to finding v'_7 in item a) in last problem we obtain that vertex 5' is not linked to vertex v (vertex v takes place of vertex v_8). Vertex 4' is linked with vertices 9 and 0 and it is not linked with vertices 3 and 5. Therefore similarly to finding v'_6 in item a) in last problem we obtain that vertex 4' is not linked to vertex v (vertex v takes place of vertex v_8). Vertex 2' is linked with vertices 4' and 5' (vertices 4' and 5' are opposite to vertices 4 and 5 in new G_{10}) and it is not linked with vertices 3 and 9. Therefore similarly to finding v'_1 in item a) in last problem we obtain that vertex 2' is not linked to vertex v (vertex v takes place of vertex v_8). Contradiction.

4) Let 3+ 5+ 7- 9- 8-. So vertices v and v_i are linked if and only if vertices v_4 and v_i are linked. Consider a new G_{10} which is obtained from the old G_{10} by replacing the vertex v_4 to v.

Vertex 9' is linked with vertices 7 and 6 and it is not linked with vertices 8 and 5. Therefore similarly to finding v'_7 in item a) in last problem we obtain that vertex 9' is not linked to vertex v (vertex v takes place of vertex v_4). Vertex 8' is linked with vertices 5 and 6 and it is not linked with vertices 7 and 9. Therefore similarly to finding v'_6 in item a) in last problem we obtain that vertex 4' is not linked to vertex v (vertex v takes place of vertex v_4). Vertex 2' is linked with vertices 9' and 8' (vertices 9' and 8' are opposite to vertices 9 and 8 in new G_{10}) and it is not linked with vertices 7 and 5. Therefore similarly to finding v'_1 in item a) in last problem we obtain that vertex 2' is not linked to vertex v (vertex v takes place of vertex v_4). Contradiction.

5) Let 3+ 5+ 7- 9- 8+

1 5 9 7 v $6'$: $6'$-

v $6'$ 8 5 0 7: $6'$+

The end.

b) Switch pair of vertices 1 and 2 and pair of vertices 1' and 2' in $2G_{10}$. Then both G_{10} become P.

c) In item a) we proved that one of uncertain edge is non-edge. Due to big amount of good renumbering of $2P$ we obtain that all of uncertain edges are non-edges. It is easy to see that complement to $2P$ is $2P$ again. Consider the complement to G. It is easy to see that it satisfies condition but all uncertain edges of $2P$ are edges in the complement to G. Contradiction.