Excircles and Dozens of Points.

(Presented by V. Filimonov and A. Zaslavsky.)

On the Origin of this Series of Problems.

The work on this series started from the problem posed by D. Tereshin.

Problem (D. Tereshin). Consider triangle ABC and its excircles: one of them touches the side AC at K and touches the lines AB and BC at L and M, the other touches the side AB at P and touches the lines AC and BC at Q and R. Prove that the intersection point X of the lines LM and QR lies on the altitude (passing through A) of the triangle ABC.

At once the geometrical solution was not found, only calculations work. Some observations made this problem more exciting. It appears that the intersection point Y of lines KM and PR lies on the altitude of triangle ABC, and the length of the segments AY and AX equal to the radii of incircle and excircle touching the side BC. Some other results were obtained, and some connections with known problems from olympiads were established. In search for the geometrical explanation of these results we tried to consider in details the touch points of the sides with incircle and excircles, and the lines joining these touch points.

The incircle and the excircles have some different properties (for example, the incircle is smaller than any of the excircles, the incircle lies inside the triangle while the excircle lies outside the triangle). Nevertheless, these four circles have deep common properties: each of them touches the three sidelines of the triangle, the center of each circle is the intersection of three angle bisectors (either internal or external). So as a rule, if one of the four circles has some property, then the others have an analogous property. That is why these four circles in some sense enjoy equal rights with respect to the original triangle. To understand some important geometrical results we need to consider all the four circles simultaneously. Thus we introduce some non-regular but symmetrical notation (see below).

The sections A, B, C of the project were made by the author of the text jointly with I. Bogdanov, the section D was added by A. Zaslavsky. Also A. Akopyan, D. Prokopenko, and V. Protassov had made many useful notes and additions.

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Notation.

In a fixed non-equilateral triangle ABC let us denote:
R, p — the radius of the circumcircle and semiperimeter;
a, b, c — lengths of BC, CA, AB;
A', B', C' — midpoints of BC, CA, AB;
AH_a, BH_b, CH_c — altitudes, H — orthocenter of triangle ABC;
Ω — circumcircle, O — circumcenter;
ω_0 — incircle, I_0 — incenter; ω_1, ω_2, ω_3 — excircles (touching segments BC, CA, AB, respectively), I_1, I_2, I_3 — centers of the excircles, r_i — radii of ω_i;
I_0', I_1', I_2', I_3' — centers of incircle and excircles of triangle $A'B'C'$.

The notation has the following symmetry: Note that 6 lines I_iI_j $(i \neq j)$ are internal and external bisectors of triangle ABC. Therefore the quadruple I_0, I_1, I_2, I_3 is orthocentric, and ABC is the orthotriangle (that is the triangle having feet of altitudes as vertices) for each of triangles $I_0I_1I_2$, $I_1I_2I_3$, $I_2I_3I_0$, $I_3I_0I_1$). To each of the points A, B, C we put into correspondence a partition of 4-element set $\{0, 1, 2, 3\}$ into two 2-element subsets: $A = I_0I_1 \cap I_2I_3$, $B = I_0I_2 \cap I_1I_3$, $C = I_0I_3 \cap I_1I_2$.

Also see the further notation

Series A: The First Dozen: Touch Points

Let A_i, B_i, C_i $(i = 0, 1, 2, 3)$ be touch points of ω_i and lines BC, CA, AB, respectively (see 12 red points in Fig. A).

Prove the following statements.

A1. A_0 и A_1 (and also A_2 and A_3) are symmetric with respect to A', moreover, $A_0A_3 = A_1A_2 = c$, $A_0A_2 = A_1A_3 = b$, $A'A_0 = A'A_1 = \dfrac{|b-c|}{2}$, $A'A_2 = A'A_3 = \dfrac{b+c}{2}$. (Similarly there is symmetry with respect to B' and C'.)

A2. a) AA_i, BB_i, CC_i are concurrent.

б) AA_1, BB_2, CC_3 are concurrent. (Similarly, triples of lines AA_0, BB_3, CC_2; AA_2, BB_1, CC_0; AA_3, BB_0, CC_1 are either concurrent or parallel.)

A3. Radical axis of pairs ω_i and ω_j are internal and external bisectors of triangle $A'B'C'$. (Find the radical centers of triples of circles ω_0, ω_1, ω_2, ω_3.)

A4. In the set of circles touching ω_1, ω_2, ω_3 there exist three circles passing through I_0'. (Formulate and prove the similar statement for the other triples of circles.)

A5. $AA_1 \parallel I_0A'$ (similarly $AA_0 \parallel I_1A'$, $AA_2 \parallel I_3A'$, $AA_3 \parallel I_2A'$, etc.).

A6. I_0A_1, I_1A_0, I_2A_3, I_3A_2 are concurrent. Determine the intersection point of these lines.

Series B: The Second Dozen: "Foci"

Let us denote $B_{01} = B_{10} = A_0B_0 \cap A_1B_1$, $B_{23} = B_{32} = A_2B_2 \cap A_3B_3$. (Here $A_0B_0 \cap A_1B_1$ is B_{01}, and not A_{01}, since A corresponds to the partition of $\{0,1,2,3\}$ into pairs $0,1$ and $2,3$.) Similarly define all 12 points: A_{ij} with $i \in \{0,1\}$, $j \in \{2,3\}$ (we put $A_{ij} = A_{ji}$); B_{ij} with $i \in \{0,2\}$, $j \in \{1,3\}$; C_{ij} with $i \in \{0,3\}$, $j \in \{1,2\}$. (See 12 violet points in Fig. B1, B2.)

Prove the following statements.

B0. $\angle B_2B_{23}B_3 = 90^\circ$ (similarly for the other angles).

B1. B_{23}, C_{23}, A_2, A_3 are concyclic.

Find the center of the circle passing through these points. (Similarly, B_{01}, C_{01}, A_0, A_1 are concyclic, etc., thus red and violet points belong to 6 circles.)

B2. A_{ij} lies on the midline $B'C'$

(similarly, B_{ij} and C_{ij} lie on midlines, thus 12 violet points lie on 3 lines $A'B'$, $B'C'$, $C'A'$).

B3. A_{13} (the same for A_{02}, B_{01}, B_{23}) lie on the circle with diameter AB,

(moreover, $A_{02}B_{01}A_{13}B_{23}$ is a rectangle.)

(Thus 12 violet points lie on 3 circles with diameters BC, CA, AB).

B4. Find the lengths $A_{02}A_{03}$, etc., in terms of a, b, c.

B5. A_{ij} lies on I_iI_j, moreover, A_{ij} is the projection of A to I_iI_j. (thus 12 violet points belong to 6 bisectors of triangle ABC).

B6. A_{02} and C_{02} are *foci* of ω_0 and ω_2 (*Foci* means that A_{02} and C_{02} is a pair of points inverse to each other with respect to each of two circles). (Thus 12 violet points are partitioned into 6 pairs of foci; in particular, from that it follows that each of ω_i contains exactly 3 violet points).

B7. Determine the radical centers of triples of circles from Problem B1 having distinct centers.

B8. Six points $A_{03}, A_{02}, C_{02}, C_{23}, B_{23}, B_{03}$ lie on a circle (also there exist 3 circles constructed in the same manner). Determine the centers of these circles. Find the radii of these circles in terms of a, b, c.

B9. A_{02} and A_{13} are either the centers of incircle and excircle or the centers of excircles, for the triangle $B'H_aH_b$.

Series C: The Third Dozen: Intersections on the Altitudes

Let $A_{(3)} = A_0C_0 \cap A_1B_1$ (Here $A_0C_0 \cap A_1B_1$ is $A_{(3)}$, and not $A_{(2)}$, since C corresponds to the partition of $\{0,1,2,3\}$ into pairs $0,3$ and $1,2$, here 3 belongs to the pair containing 0). Similarly, $A_{(2)} = A_0B_0 \cap A_1C_1$, $A_{(0)} = A_2B_2 \cap A_3C_3$, $A_{(1)} = A_2C_2 \cap A_3B_3$; in the same manner define $B_{(i)}$ and $C_{(i)}$ — totally 12 green points in Fig. C.

Prove the following statements.

C1. Points $A_{(i)}$ lie on the line AH_a (similarly for $B_{(i)}$ and $C_{(i)}$, thus 12 green points lie on the altitudes of triangle ABC).

C2. The length of $AA_{(i)}$ is equal to r_i.

C3. $A_{(i)}A_i$ is parallel to one of two bisectors of angle A.

C4. Prove that $A_{(1)}A_1$, $B_{(2)}B_2$, and $C_{(3)}C_3$ are concurrent. (Similarly, there exist three triples of concurrent lines: $A_{(0)}A_0$, $B_{(3)}B_3$, $C_{(2)}C_2$; $A_{(3)}A_3$, $B_{(0)}B_0$, $C_{(1)}C_1$; $A_{(2)}A_2$, $B_{(1)}B_1$, $C_{(0)}C_0$.)

C5. Triangles $A_1B_2C_3$ and $A_{(0)}B_{(0)}C_{(0)}$ are symmetric (with respect to a point). Define the center of symmetry. (Similarly, pair of triangles $A_0B_3C_2$ and $A_{(1)}B_{(1)}C_{(1)}$, $A_3B_0C_1$ and $A_{(2)}B_{(2)}C_{(2)}$, $A_2B_1C_0$ and $A_{(3)}B_{(3)}C_{(3)}$ are symmetric.)

C6. Triangles $A_{(1)}B_{(2)}C_{(3)}$, $A_{(0)}B_{(3)}C_{(2)}$, $A_{(3)}B_{(0)}C_{(1)}$, $A_{(2)}B_{(1)}C_{(0)}$ Have a common circumcenter (thus green points lie on 4 concentric circles). Define the common circumcenter.

C7. Find AH, BH, CH in terms of radii r_i.

C8. Find the radius of the circumcircle of triangle $A_{(1)}B_{(2)}C_{(3)}$ in terms of R and r_0. (Similarly, find the radii of the circles from Problem C6.)

C9. I_iA' passes through $A_{(i)}$ (similarly, I_iB' passes through $B_{(i)}$, I_iC' passes through $C_{(i)}$).

C10. Let l_a be a line passing through A and parallel to BC. $M = A_0C_0 \cap l_a$, $N = A_0B_0 \cap l_a$. Prove that $A_{(0)}$ is the circumcircle of the triangle A_0MN.

Determine the orthocenter of the triangle A_0MN.

Series D: The Fourth Dozen.

Let $C_0^* = A_0B_0 \cap A_1B_2$, and similarly define 12 blue points A_i^*, B_i^*, C_i^* (see Fig. D). (The description of these points is as follows: Take one of the circles, for example ω_0. Take its two touch points, say A_0, B_0. Take the touch points of these sides with two other circles that are symmetric to A_0, B_0 with respect to the midpoints of the sides — A_1, B_2. Take the intersection points of the lines joining pairs of these points.)

Prove the following statements.

D1. The sidelines of triangle $A_i^*B_i^*C_i^*$ pass through the vertices of triangle ABC.

D2. A line passing through C_i^* intersects BC, AC at A'', B'', respectively. Show that $A''B_i^*$ and $B''A_i^*$ intersect at some point C'' of the line AB.

D3. AA'', BB'', CC'' have a common point that is isogonally conjugate to some point of the line OI_i.

D4. The circumcircle of triangle $A''B''C''$ passes through the Feuerbach point F_i.

D5. Four blue points denoted by the same letter lie on a sideline of the orthotriangle.

D6. a) Triangles $A_i^*B_i^*C_i^*$ and ABC are perspective (i.e. the lines joining corresponding vertises of these triangles are concurrent)

b)Try to find some relations between four centers of perspective.

D7. (The generalization of the problem D4) Let C^{**} be a point on line H_aH_b. An arbitrary line passing through C^{**} intersects BC, AC at A'', B'', respectively. Let P be the point of intersection of lines AA'' and BB'', and C'' be the point of intersection of lines CP and AB. Then the circumcircles of all triangles $A''B''C''$ have the common point.

Tasks from series A, B, C marked # and also from series D were given to the paticipants after the intermediate finish.