Magic graphs

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Definitions and notations

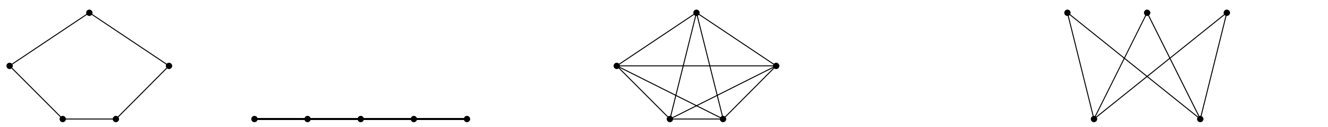
All the graphs under consideration are supposed to be without isolated vertices, multiple edges and loops.

The words "cycle" and "path" mean *simple* cycle and *simple* path in a graph.

For every edge of a graph we assign a positive number that we call a weight of this edge. A graph is called *semimagic* if it is possible to choose weights of its edges and a positive number s such that for each vertex the sum of weights of its edges equals to s. A graph is called *magic* if it possible to choose these weights to be pairwise different. Observe that a vertex of degree 1 in the semimagic graph is necessarily the endpoint of the isolated edge. A magic graph can contain at most 1 isolated edge.

A subgraph F of a given graph G is called a *skeleton*, if it contains all the vertices of G and none of them is isolated vertex in F. *1-2-skeleton* is a skeleton such that all its vertices have degree 1 or 2 and for each component the degrees of its vertices are the same. In other words 1-2-skeleton consists of isolated edges and simple cycles only. For each 1-2-skeleton we can split all the edges of the graph onto 3 groups: edges that belong to the *cyclic* part of F (we will denote it by F_c); edges that belong to the *linear* part of F, i. e. isolated edges in F (we will denote it by F_ℓ); and edges that do not belong to F. We say that 1-2-skeleton *separates* edges e_1 and e_2 if these two edges belong to different groups. In other words at least one of them belongs to F but at most one belongs to F_c and at most one belongs to F_ℓ.

We will use the following notations: C_n is the cycle with n edges ($n \geqslant 3$); P_n is the path with n edges; K_n is the complete graph with n vertices; $K_{m,n}$ is the complete bipartite graphs with parts of m and n vertices.



A cycle C_5 | A path P_5 | A complete graph K_5 | A complete bipartite graph $K_{2,3}$

Figure 1: Some standard graphs

A direct product $F \times G$ of two graphs is the following graph. Its vertex set is the set of all pairs (v, w), where v is a vertex of F, w is a vertex of G. The vertices (v_1, w_1) and (v_2, w_2) are joined by an edge, if either $v_1 = v_2$ and G contains the edge w_1w_2, or $w_1 = w_2$ and F contains the edge v_1v_2. The graph $G \times P_1$ is called *the double* of graph G. *Dum-bell* is a graph consisting of either two odd cycles which share exactly one common vertex, or two odd cycles joined by a path of an arbitrary length.

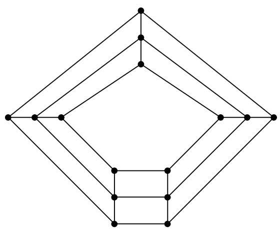


Figure 2: Graph $C_5 \times P_2$

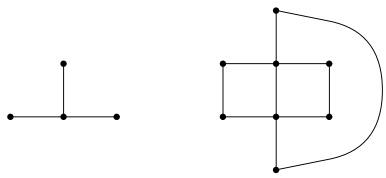


Figure 3: Graph and its double

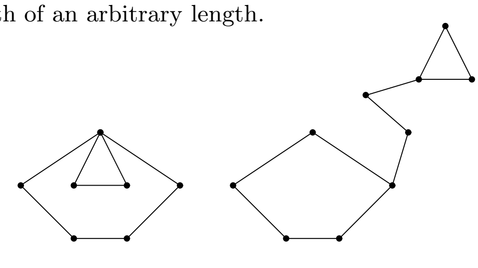


Figure 4: Dum-bells

1 *Examples*

1.1. Show that magic graphs with less than 5 vertices do not exist, except the graph P_1 (one edge).

1.2. Prove that a bipartite graph with odd number of vertices is non magic. Could it be semimagic?

1.3. Determine wether these graphs are semimigic or magic (the answers may depend on n and m)
a) K_n; b) $K_{m,n}$; c) $P_n \times P_1$; d) $P_n \times P_m$ при $n, m > 1$; e) $C_n \times P_1$; f) $C_n \times P_m$, $n \geqslant 3$, $m > 1$; g) cycle of $2n$ vertices, where every two opposite vertices are joined by edge.

2 *Semimagic graphs*

2.1. Prove that if a semimagic graph G contains an even cycle then G contains also a semimagic skeleton (i. e. the skeleton which is a semimagic graph itself) such that not all the edges of the cycle belong to this skeleton.

2.2. Prove that if a semimagic graph G contains a dum-bell then G contains also a semimagic skeleton such that not all the edges of the dum-bell belong to this skeleton.

2.3. Prove that each semimagic graph has 1-2-skeleton.

2.4. The main theorem about semimagic graphs. Prove that a graph is semimagic if and only if each of its edges belongs to some 1-2-sceleton.

In the following problems we find out when a graph contains 1-2-skeleton. We call a graph *soft* if it does not have 1-skeleton, and *solid* if it contains 1-skeleton. A soft graph is called *saturated* if it turns solid when an arbitrary edge has been added.

Let G be an arbitrary graph, S is an arbitrary set of its vertices. Denote by $G \setminus S$ the graph obtained by deletion of all the vertices of the set S and its edges.

2.5. Let G be a saturated soft graph, S be the set of all its vertices such that each of them is joined with all other vertices. Prove that all components of the graph $G \setminus S$ are complete graphs.

2.6. The main theorem about saturated soft graphs. A graph G is saturated and soft if and only if either

a) G is a complete graph with odd number of vertices, or

b) the number of vertices of G is even and we can split it onto complete graphs S_0, G_1, G_2, $\ldots$, G_k, where $k = |S_0| + 2$, such that for all i the number of vertices in G_i is odd and every vertex of G_i is joined with all the vertices of S.

2.7. Prove that a graph G is solid if and only if for each set S of vertices of G the graph $G \setminus S$ has at most $|S|$ odd components.

2.8. Prove that graph G contains 1-2-skeleton if and only if for each set S of vertices of G the graph $G \setminus S$ has at most $|S|$ isolated vertices.

3 Magic graphs

3.1. Prove that each magic graph has the following two properties:

(1) Every edge of the graph belongs to some 1-2-skeleton.

(2) Every two edges are separated by some 1-2-skeleton.

3.2. Prove the converse statement: if a graph has these two properties then it is magic.

3.3. Graph G' is obtained from magic graph G by adding a new edge and this new edge belongs to some 1-2-skeleton of graph G'. Prove that G' is magic.

3.4. Graph G consists of two (non isomorphic) components, each component has at least 3 vertices. Both components are magic graphs. Is it true that G is necessarily magic?

3.5. a) For each edge e in a semimagic graph G (without isolated edges) there exists a 1-2-skeleton, whose cyclic part does not contain e. Prove that the double of G is magic.

b) G is a semimagic graph without isolated edges, H is an arbitrary connected graph without isolated edges. Prove that $G \times H$ is a magic graph.

3.6. G is an arbitrary graph with at least 4 vertices. Graph G' is obtained by adding one more vertex to G, and this vertex is joined with all the "old" vertices of G. Prove that the graph G' is magic if and only if the graph G is without isolated edges and it has 1-2-skeleton.

3.7. a) Graph G has $n \geqslant 5$ vertices. The degrees of vertices of G are at least $\frac{n}{2} + 1$. Prove that g is a magic graph.

b) Prove that for any large n there exist non semimagic graph such that the minimal degree of its vertices equals to $[n/2]$.

3.8. G is a connected magic graph with $n \geqslant 5$ vertices and r edges. Prove that $r > \frac{5}{4}n$.

3.9. For $n = 5$, 6, 7, 8 construct a connected magic graph with n vertices and r edges, where r is the minimal integer that satisfies the inequality $r > \frac{5}{4}n$.

3.10. Construct an analogous graph for each $n \geqslant 5$.

3.11. Prove that there exists a connected magic graph with n vertices and r edges, if the pair (n, r) satisfies the inequality $\frac{5}{4}n < r \leqslant \frac{n(n+1)}{2}$.

Semifinal

4 Regular graphs

We will not discuss when regular graphs of degree 1 and 2 are magic. Below we will consider regular graphs of degree at least 3.

A pseudocycle is an even cycle or dum-bell (remind that both cycles in dum-bell are even).

Consider an even cycle. Put alternatively on its edges weights 1 and -1, let all other edges have weight 0. We say that two edges are *weakly separated* by the cycle if they have had different weights. Analogously, for each dum-bell, put the weights ± 1 and ± 2 on its edges as in fig. 5 (where $a = 1$), and let all other edges have weight 0. We say that two edges are *weakly separated* by the dum-bell if they have had different weights. Finally, we say that two edges are *weakly separated* by a pseudocycle if there exists an even cycle or a dum-bell that weakly separates these edges.

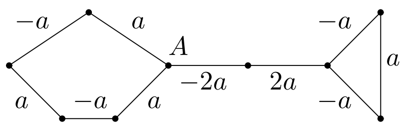


Figure 5: Alternative weights of dum-bell edges

4.1. Prove that every edge of the regular graph of degree $d \geqslant 3$ belongs to some pseudocycle.

4.2. Prove that the regular graph of degree $d \geqslant 3$ is magic if and only if any two of its edges are separated by pseudocycle.

4.3. Prove the following theorem. Let G be a regular graph of degree $d \geqslant 3$ and $G_1, \ldots, G_k$ be its components. Then G is magic if and only if all G_i are magic.

Index of edge connectivity $\ell(G)$ is the minimal number of edges of G that should be erased in order to obtain disconnected graph.

4.4. Let G be connected regular bipartite graph. Prove that the property "to be magic" or "to be non-magic" depends on $\ell(G)$ only and completely investigate this dependance.

5 Addendum

5.1. To the problem 1.3.a. Graph is called *supermagic* if its magic weights are consecutive positive integers.

For which n graph K_n is supermagic?

5.2. To the problem 3.7. A graph has 2009 vertices of degree at least 1006. At most 500 edges were deleted. Prove that the rest graph is still magic.

Solutions

1 *Examples*

1.1. If a graph with 4 vertices has 1 or 2 edges then it has isolated vertex. If it has 3 or 4 edges then it contains two adjacent vertices of degree 2 and hence it is non-magic. The graph with 6 edges is necessarily K_4, see problem 1.3a).

Finally, if it has 5 edges then it is isomorphic to the cycle $ABCD$ with the diagonal AC. Then the sum of weights of edges adjacent to vertices A and C equals $2s$. Geometrically, it is the sum of weights of all edges, where the weight of AC has multiplicity 2. The other way to obtain the sum $2s$ is to sum up the weights of edges adjacent to vertices B and D. This is a sum of all edges of the graph except AC. Therefore AC has zero weight, which is forbidden.

1.2. Answer: the graph is not semimagic. Let one part of the graph contains k vertices, the second part contains ℓ vertices, and let s be the sum of weights of all edges adjacent to the same vertex. If the graph is semimagic, then the sum of weights of edges adjacent to vertices of the first part equals ks, the sum of weights of edges adjacent to vertices of the second part equals ℓs, and both sums equals the sum of weights of all edges of the graph. Therefore $\ell = k$. This is impossible because the total number of vertices is odd.

1.3. a) Answer: the graph is always semimagic, it is magic for $n = 2$ and $n > 5$ only.

To show that the graph is semimagic take all weights equal to 1.

If $n = 3$ the graph is not magic, it is evident.

If $n = 4$ we have 4 vertices A, B, C, D. Assume that it is magic, let s be the sum of weights of all edges adjacent to the same vertex. Then $2s$ is the sum of all edges adjacent to vertices A and C, i.e. the sum of all edges of the graph except CD but with weight of AC counted twice. By the analogous consideration for vertices B and D we will obtain that the weights of AC and BD coincide.

If $n > 5$ the graph is magic. We will prove this by the following construction. Since the graph is regular, we may consider arbitrary weights (not necessarily positive), because in the regular graph we can make all the weights to be positive by adding a large positive constant. Let us describe the *main construction* of magic weights for the regular graphs by means of even cycles.

> Write out all even cycles that are contained in our graph and enumerate them by numbers from 1 to N (where N is the total number of these cycles). For any k put the weights $\pm 3^k$ alternatively on the edges of k-th cycle. After that for each edge sum up all the weights on it.

Let us check that this set of weights is magic. Indeed, the sum of weights of edges adjacent to every vertex equals 0, because the contribution of each cycle to this sum is 0. Now let us check that all weights are distinct. For each edge of the graph write out the list cycles which contain this edge. It is clear that for any two edges there exists an even cycle that contains one of these edges only. Therefore for any two edges their lists of cycles do not coincide. But then the sums of weights determined by the cycles are not equal. This is because the weights obtained by our construction may be regarded as N-digital ternary numbers in system (with base 3) with digits 0 and ± 1. Since all lists are distinct then all these ternary numbers are distinct also.

b) Answer: the graph is semimagic for $m = n$ only. For $m = n > 2$ it is magic.

The equality $m = n$ is necessary for the graph to be semimagic (see problem 1.2).

If $m = n$ the graph is regular and hence semimagic. If $m = n = 2$ it is evidently non-magic (see problem 1.1). And if $m = n > 2$, the graph is magic, due to construction from the previous solution.

c) Answer: the graph is semimagic but non-magic.

To show that it is semimagic it is sufficient to assign weights of all edges to be 1, except leftmost and rightmost edges of weight 2. The graph is non-magic because it contains adjacent vertices of degree 2.

d) Answer: the graph is magic if either m or n is odd. If both m and n are even, then the graph is not semimagic.

If both m and n are even, then the graph is bipartite with $(n + 1)(m + 1)$ vertices (odd number). This graph is not semimagic due to problem 1.2.

Now prove that the graph is magic for odd n and $m > 1$.

At first consider case $m = 2$. Consider the initial placement of nonnegative semimagic weights depicted on fig. 6: bold edges have weight $2m$, dashed edges have weight 0, all other edges have weight M, where M is a big number, that we will choose later. This set of weights is almost semimagic, but some weights here are zero and the graph is not regular.

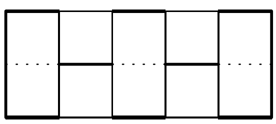


Figure 6: Almost semimagic weights on graph $P_n \times P_2$

Now we will perform the main construction of magic weights for the regular graphs by means of even cycles, but with 3 corrections (because our graph is not regular):

1) In the main construction we will consider 4-cycles only (i.e. the sides of cells).

2) The final weight of an edge will be equal to the sum of its initial weight and the weight obtained by the main construction.

3) When we assign positive and negative weights of edges in cycles, we choose plus sign for edges whose initial weight has been equal to zero.

Finally, choose M so big that all the final weights turn out to be positive and distinct. Then this set of weights will be magic.

Now consider a case n is odd, $m > 2$. We perform analogous actions. The initial weights are depicted on fig. 7 (odd number n corresponds to the vertical side of the picture).

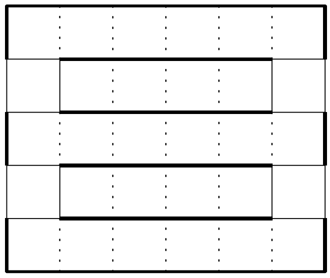


Figure 7: Almost semimagic weights on graph $P_n \times P_m$

e) Answer: the graph is always semimagic. It is magic for even n only.

We may realize this graph as edges of n-gonal prism.

Let $n = 2k$. The graph contains evident cycles of length 4 (sides of facets) and two $2k$-cycles (sides of bases). The property "for any two edges there is a cycle that contains one of them only" is satisfied. Therefore we can perform the main construction for regular graphs.

Let us prove that the graph is non-magic for $n = 2k + 1$. As usual let s be the sum of weights of edges adjacent to the same vertex. It is easy to see that the sum of weights of all edges equals nd. Denote our prism by $A_1A_2 \dots A_{2k+1}B_1 \dots B_{2k+1}$. The sum of weights of edges adjacent to vertices A_1, A_3, $\dots$, A_{2k+1}, B_2, B_4, $\dots$, B_{2k}, equals nd and can be interpreted as the sum of weights of all edges of the graph except B_1B_{2n+1} and with edge A_1A_{2n+1} counted twice. Hence, the weights of A_1A_{2n+1} and B_1B_{2n+1} are equal.

f) Answer: the graph is magic.

Solution is analogous to solution 1.3d). Let A_1, $\dots$, A_{m+1} be vertices of graph P_m. Initial placement of nonnegative weights on the graph looks as follows: all edges of subgraphs of the form $C_n \times A_i$ have weight 2, all other weights are 0. We apply the main construction for 4-cycles only.

g) Answer: the graph is always semimagic; it is magic for odd n.

It is semimagic because it is reguar. For $n = 2$ this graph is K_4, we discuss it in problem 1.3 a). For odd n it is magic because the main construction works (we have a good store of even cycles here: 4-cycles that contains subsequent diameters and $(n + 1)$-cycles of the form "semicircle").

For even n the graph is non-magic. Let $A_1A_2 \dots A_nB_n \dots B_2B_1$ be vertices of the given cycle. The set of edges adjacent to all vertices A_i, B_i, where i runs over odd numbers, is the set of all edges of the graph except A_nB_n and with A_1B_1 counted twice. It follows that in any semimagic set of weights the opposite edges of the cycle have the same weight.

2 Semimagic graph

2.1. Let a be the minimal weight of the edges in the given cycle. We will move along the cycle and decrease and increase by a alternatively the weights of the edges of the cycle. After that erase all the edges with zero weight. The remaining graph together with the weights of its edges will be desired semimagic skeleton.

2.2. Let A be the vertex of degree 3 (or 4) of one of odd cycles of the dum-bell. We will bypass the dum-bell starting from the vertex A. First of all we will move along the odd cycle and assign the weights to its edges to be $\pm a$ alternatively. After return to the vertex A, we have two edges of weight a adjacent to A. Then we move along the handle of dum-bell and assign the weights of its edges to be $\mp 2a$ alternatively. After that we move along the second cycle assigning its edges weights $\pm a$ alternatively. We obtain semimagic weights with $s = 0$ (see fig. 5).

Now we are going to add new weights to old ones. For this choose the value of parameter a so that all the weights of the graph would be nonnegative and the weight of at least one of the edges would be equal to 0. We will obtain semimagic weights. After erasing all the edges with zero weight we obtain the desired skeleton.

2.3. The constructions of solutions 2.1, 2.2 allow us to decrease consequently the number of edges in the graph by destroying its even cycles and dum-bells. The graph will be semimagic during all these operations and therefore there will be no pendant vertices in it (except the endpoints of isolated edges). Observe that if a component of the graph contains two even cycles then it contains also an odd cycle or a dum-bell. And if a component contains exactly one (odd) cycle and does not contain pendant vertices then this component is exactly this odd cycle.

So, after destroying all even cycles and dum-bells we will obtain a graph consisting of several isolated edges and several (odd) cycles.

2.4. Let us check that every edge of a semimagic graph belongs to some 1-2-skeleton. Let G be a graph with minimal number of edges such that one of its edges, say, e does not belong to any 1-2-skeleton. Fix a set of semimagic weights $\mathcal{W}$ on graph G. Fix an arbitrary 1-2-skeleton and construct one more semimagic set of weights $\mathcal{S}$ as follows. Let each edge of the linear part of the skeleton has weight a, each edge of the cyclic part has weight $a/2$ and all other edges (including e) have weight 0. Choose the value of a so that all the weights of the set $\mathcal{S}$ do not exceed the corresponding weights in the set $\mathcal{W}$ and for at least one edge we have an equality. Now subtract from weights of $\mathcal{W}$ the weights of $\mathcal{S}$. We obtain a semimagic set of weights, where not all the weights are equal to 0, because the weight of edge e has not changed. Now remove all edges of zero weight. We obtain semimagic graph G' which is a skeleton of G. G' contains less edges than G and edge e does not belong to any 1-2skeleton of G', because "the skeleton of my skeleton is my skeleton". This is a contradiction with the definition of G. Therefore the graph G does not exist.

Now let us check that if every edge of the graph belongs to some 1-2-skeleton, then the graph is semimagic. For each 1-2-skeleton assign the weight of its cyclic edges be 1 and the weight of its isolated edges be 2. We obtain semimagic set of weights (but with zero weights). Let us sum up all these set of weights over all 1-2-skeletons. The result is the desired semimagic set of weights.

2.5. We take this problem and the following solution from [1, § 3.1.2]. Let A, B, C be vertices of $G \setminus S$ and B is joined with both A and C. It is sufficient to prove that graph G contains edge AC. Assume that this is not true. By the definition of the set S graph G contains the vertex D such that the edge BD does not belong to the graph. If we add edge AC to graph G, then the new graph has 1-skeleton (since graph G is saturated). It is clear that edge AC must belong to this skeleton. Color this skeleton in red. Analogously, if we add edge BD to graph G, we can find blue 1-skeleton containing edge BD. Now from these two skeletons we will construct a 1-skeleton of graph G and get a contradiction.

Consider the union of these skeletons; the edges which are red and blue simultaneously we will consider as usual (non-multiple) edges. Then this union is a 1-2-skeleton of graph $G \cup AC \cup BD$, and all its cycles are even because red and blue edges alternate.

It is clear that edges AC and BD are both in the cyclic part. If these edges belong to different cycles, then the desired 1-skeleton can be constructed as follows. Take the red skeleton and replace all red edges of the cycle that contains AC by blue edges of the same cycle. Now consider the second case, let edges AC and BD belong to cycle γ. Let us bypass cycle γ starting from vertex B and edge BD till we reach vertex A or C. Let it be A for definiteness. Since the red edge of vertex A is AC, we finish our movement by blue edge. Hence the path from B to A starts and finishes with blue edges. Take the blue skeleton, replace all blue edges of the path by red edges of the same path and add edge AB. We obtain 1-skelton of graph G.

2.6. We take this problem and the following solution from [1, § 3.1.2]. It is evident that a soft saturated graph with odd number of vertices is necessarily complete. Let the number of vertices in G be even; let S be the set of all vertices of G that are joined with all other vertices and s be the number of these vertices; let G_1, G_2, …, G_k be components of connectivity of graph $G \setminus S$. Due to the statement of the previous problem we know that they are all complete graphs.

If $G \setminus S$ has at most s odd components, construction of the 1-skeleton is trivial. Assume that $G \setminus S$ has at least $s+1$ odd components; taking into account parity of number of vertices of G, we conclude that $G \setminus S$ has at least $s+2$ odd components. If the number of odd components is greater than $s+2$, join any two of them by an edge. We obtain graph G_1 such that graph $G_1 \setminus S$ has more than s odd components of connectivity. There are no 1-skeletons in this graph (it is evident, it follows also from the easy part of the statement of problem 2.7), but this is impossible because graph G is saturated.

Thus, graph G has exactly $s+2$ odd components. It can not have even components due to analogous reasons.

2.7. This statement is classical Tutte theorem. The following proof is from [1, § 3.1.2].

If we can find the set of vertices S in graph G, such that graph $G \setminus S$ has more than $|S|$ odd components of connectivity, then graph G is soft. It is clear.

Check the converse statement. Assume that for any subset S of the set of vertices of G graph $G \setminus S$ has at most $|S|$ components of connectvity but at the same time graph G is soft.

The number of vertices of graph G is even because otherwise $S = \varnothing$ leads to the contradiction. Add several edges to graph G to obtain soft saturated graph G'. Let S' be set of vertices joined with every vertex of G', s be number of its elements. Since G' and G have equal (even) number of vertices due to main theorem about soft saturated graphs we have that graph $G' \setminus S'$ contains $s+2$ odd components, each of them is a complete graph. Now remove those edges we have add making graph saturated. It is possible that some components of graph $G' \setminus S'$ will fall to parts but at least one part of odd component will be odd and the total number of odd components will be grater than s. Thus, the set S' disproves the property of G under consideration.

2.8. Let n be the number of vertices of graph G. Construct a new graph G' with $2n$ vertices. For every vertex v in G take two vertices v' and v'' in G'; for every edge uv define two edges in G': $u'v''$ and $u''v'$ Then G' is a bipartite graph, that has twice as many edges as G.

Remark that the existence of 1-2-skeleton in G is equivalent to the existence of perfect matching in G'. Indeed, for each cycle $v_1v_2 \ldots v_\ell$ of the skeleton graph G' has edges $v_1'v_2''$, $v_2'v_3''$, …, $v_\ell'v_1''$; analogously for any isolated edge graph G' contains edges $u'v''$ and $v'u''$. It is clear that all these edges form a perfect matching. Conversely, for any perfect matching of graph G' it is not difficult to construct a 1-2-skeleton. For example, the edges $u'v''$, $v'w''$, $w'z''$, $z'u''$ of the perfect matching determine a cycle $uvwz$ of the graph G, and edges $u'v''$ and $v'u''$ determine an isolated edge uv of the skeleton.

As we know, for each set S of vertices of G the graph $G \setminus S$ has at most $|S|$ isolated vertices. Let us reformulate this property in terms of graph G'. Consider an arbitrary set S of vertices of graph G. What does it mean that after deletion of this set the vertex u becomes isolated? This means that in the graph G' all neighbours of u' belong to S''. If after removing set S we have $k > |S|$ isolated vertices then the conditions of Hall theorem is not satisfied in graph G because we found a set of k vertices that has at most $|S|$ neighbours (the last number is less then k). The converse is also true (i.e. if the conditions of Hall theorem is not satisfied, then the property under discussion holds). Therefore this property is equivalent to the conditions of Hall theorem in graph G' that is equivalent to existence of perfect matching in G'.

3 Magic graphs

3.1. (1) All the semimafic graphs have this property.

(2) We will prove more general fact: if a semimagic graph has a semimagic set of weights such that two edges, say e_1 and e_2, have distinct weights, then the edges e_1 and e_2 are separated by a 1-2-skeleton.

It can be done analogously to the solution of problem 2.4. Choose a minimal graph; fix a set of weights, where not all weights are equal; subtract by a suitable way the weights belonging to a skeleton; we will obtain smaller graph. Since the initial graph was minimal, one of edges e_1, e_2 must receive zero weight and should be removed. In the remaining graph the second edge due to statement of problem 2.4 belongs to some 1-2-skeleton, that sepapates these edges in the initial graph.

3.2. Let us enumerate all the 1-2-skeletons. Let the edges of cyclic part of k-th skeleton have weights 3^k, and edges of linear parts have weight $2 \cdot 3^k$. For each edge sum up all its weights over all 1-2-skeletons. The set of weights obtained is semimagic due to uniqueness of ternary notation of numbers.

3.3. It follows from 3.2.

3.4. Answer: no, graph G is not necessarily magic. Two magic graphs are depicted on the fig. 8 For any set of magic weights the edges denoted by dashed lines must have weight $s/2$.

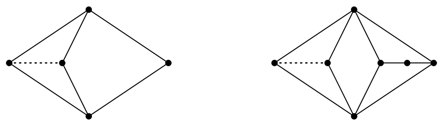


Figure 8: Union of magic graphs can be non-magic

3.5. a) The double G^2 consists of two copies G_1 and G_2 of the graph G and of the set of edges E between the corresponding vertices. The corresponding edges in parts G_1 and G_2 are called *parallel*. The edges from the set E are called *vertical*. A subgraph of G^2 consisting of two copies of some subgraph of G is called *duplicated*.

First of all describe a construction of *rotation of parallel edges* in G^2. Let subgraph H of graph G^2 be the union of two subgraphs in parts G_1 and G_2 (without vertical edges) such that these subgraphs contain parallel edges A_1B_1 and A_2B_2. Let us replace edges A_1B_1, A_2B_2 in subgraph H by edges A_1A_2 and B_1B_2. Denote the new subgraph by H'. We say that subgraph H' is obtained from H by the rotation of parallel edges. It is clear that both of H and H' are (or are not) 1-2-skeletons.

To prove that graph G^2 is magic let us apply criterion from problems 3.1–3.2.

(1) Every edge belongs to 1-2-skeleton. It is clear for edges from G_1 (and from G_2): duplicate the 1-2-skeleton containing this edge in G_1. For vertical edges choose suitable rotation of edges of appropriate duplicated 1-2-skeleton.

(2) Every two edges are separated by 1-2-skeleton.

- If both of edges e_1 and e_2 belong to G_1 consider a duplicated 1-2-skeleton containing e_1. If it does not separate e_1 and e_2, then both edges belong to the skeleton. By rotating edge e_2 and its parallel copy we obtain a separating skeleton.
- If e_1 belongs to G_1, and e_2 belongs to G_2, consider the union of 1-2-skeleton in G_1 containing e_1 (it exists due to the statement of problem 2.4), and 1-2-skeleton in G_2 that does not contain e_2 (it exists by the condition of the problem).
- If e_1 belongs to G_1 and e_2 is vertical consider a duplicated skeleton containing e_1.
- Finally, if both of edges A_1A_2 and B_1B_2 are vertical choose in G_1 an edge A_1X_1 (where $X_1 \neq B_1$) or B_1Y_1 (where $Y_1 \neq A_1$), this edge exists since G has no isolated edges. Consider a duplicated skeleton containing this edge and rotate this edge together with parallel edge.

b) Analogously to a).

3.6. We take the statement of the problem from [5]. The following solution was found by participants of the conference.

1. Check that if graph G' is magic then graph G has 1-2-skeleton and has no isolated vertices and edges. Isolated vertex in G corresponds to a pendant vertex in G'. Isolated edge in G corresponds to two adjacent vertices of degree 2. Both constructions are impossible in a magic graph.

Assume that G does not contain 1-2-skeleton.

Denote by S the new vertex of graph G'. Fix an arbitrary 1-2-skeleton in graph G', w.l.o.g. we may assume that all its cycles are odd. Consider the component of the skeleton that contains vertex S. If this component is an odd cycle, remove vertex S and split other vertices of the cycle on pairs. Together with other components of the skeleton they form a skeleton of graph G. Therefore we may assume that this component is an isolated edge SA_1. Now we will construct two sets of vertices $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ such that the edges A_iB_i ($1 \leqslant i \leqslant n$) belong to skeleton K and all the vertices adjacent to vertices of the set $\mathcal{A}$ belong to $\mathcal{B}$.

Let $\mathcal{A} = \{A_1\}$, $B_1 = S$. Assume that the sets $\mathcal{A} = \{A_1, \ldots, A_k\}$ and $\mathcal{B} = \{B_1, \ldots, B_k\}$ have constructed already and there is an edge that joins some vertex of the set $\mathcal{A}$ with some vertex outside $\mathcal{A} \cup \mathcal{B}$, say A_kB_{k+1}. Vertex B_{k+1} belongs to some component of the skeleton K. If this component is an odd cycle we can easily reconstruct the skeleton K to obtain a 1-2-skeleton of graph G.

> To do this consider the shortest path in $\mathcal{A} \cup \mathcal{B}$ from B_1 to B_{k+1} such that the vertices of $\mathcal{A}$ and $\mathcal{B}$ alternate. The path has even length, choose all its even edges (the last of them has endpoint B_{k+1}) and split onto pairs all other vertices of the odd cycle.

Therefore we may assume that vertex B_{k+1} belongs to the isolated edge $B_{k+1}A_{k+1}$ of skeleton K. Then place vertex B_{k+1} to the set $\mathcal{B}$ and vertex $\mathcal{A}_{k+1}$ to the set $\mathcal{A}$.

We will increase sets $\mathcal{A}$ and $\mathcal{B}$ by this algorithm until it is possible. As a result we obtain that the set $\mathcal{A}$ is joined by edges with $\mathcal{A} \cup \mathcal{B}$ only. Assume there exists an edge A_iA_j consider the shortest path between these two vertices such that the vertices of $\mathcal{A}$ and $\mathcal{B}$ alternate in it (the existence of this path can be easily seen from the process of construction of sets $\mathcal{A}$ and $\mathcal{B}$). This path together with edge A_iA_j form an odd cycle. Then the skeleton K can be reconstructed to the skeleton of graph G as described above.

So we have sets $\mathcal{A}$ and $\mathcal{B}$. Since these sets have equal number of elements, the sums of weights of their vertices are equal. But the sum of weights of $\mathcal{A}$ equals the sum of weights of all edges A_iB_j while the sum of weights of $\mathcal{B}$ equals the sum of weights of all edges A_iB_j and edges of the form B_1B_i (remind that $B_1 = S$ is adjacent to all other vertices of graph G). We obtain a contradiction.

2. The proof of the converse statement — if graph G has 1-2-skeleton and does not contain isolated edges, then G' satisfies conditions of the problem 3.1 (and therefore it is magic) — is not difficult technical exercise. The skeletons that we need for edge separating can be constructed by a suitable transformation of skeleton in G.

3.7. a) Prove that an arbitrary two edges e and f can be separated by some 1-2-skeleton. Remove the endpoints of edge e (and all their edges) from the graph G. The remaining part of the graph has $n-2$ vertices of degree at least $\frac{n}{2} - 1 = \frac{n-2}{2}$. Then it is known that there is a Hamiltonian cycle in this graph (the cycle that passes trough all the vertices of the graph). This cycle together with edge f forms 1-2-skeleton that separates edges e and f.

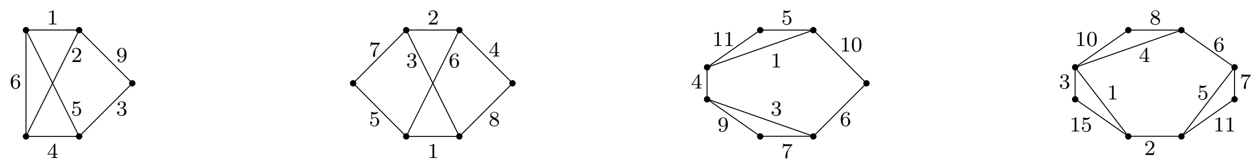
b) Consider the graph G with $n = 2k$ vertices $X_1, \ldots, X_k, Y_1, \ldots, Y_k$ such that its set of edges consists of all edges X_iY_j and edge Y_1Y_2. The degree of each vertex X_i is at least $k = \frac{n}{2}$. Let us prove that G is non semimagic.

Consider an arbitrary 1-2-skeleton of G. In this skeleton each vertex X_i has one or two adjacent vertices among the vertices Y_i. Since we have equal number of vertices of both types, the 1-2-skeleton must be perfect matching. Therefore edge Y_1Y_2 does not belong to any 1-2-skeleton. Hence graph G is non-magic.

3.8. A magic graph has no vertex of degree 1 and any two vertices of degree 2 are not adjacent in it. Let V be the set of vertices of degree 2 (possibly, $V = \varnothing$), W be the set of vertices of degree at least 3. Let s be the sum of weights in each vertex. The sum of weights of edges that have an endpoint in V equals $s|V|$. The second endpoints of these edges belong to W, therefore this sum does not exceed $s|W|$. So, $|V| \leqslant |W|$. The sum of degrees of all the vertices is at least $2|V| + 3|W|$ hence the number of edges is not less than $|V| + \frac{3}{2}|W|$. But $|V| + \frac{3}{2}|W| \geqslant \frac{5}{4}(|V| + |W|) = \frac{5}{4}n$, because $|W| \geqslant |V|$.

The equality would be possible if there are no edges with both endpoints in W, i.e. in a bipartite graph. But in this case $s|V| = s|W|$, so $|V| = |W|$. Then the number of edges between V and W equals $2|V|$ (from the point of view of the set V) and in the same time it is at least $3|W|$. Hence $|V| \geqslant \frac{3}{2}|W|$ that is impossible. The inequality $r > \frac{5}{4}n$ is proven.

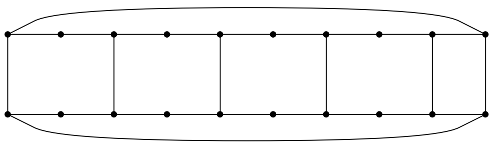
3.9. See fig.9.



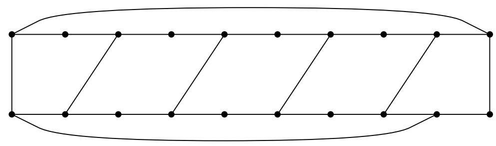
a) 5 vertices, 7 edges b) 6 vertices, 8 edges c) 7 vertices, 9 edges d) 8 vertices, 11 edges

Figure 9: Minimal magic graphs

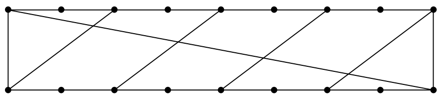
3.10. Magic graphs with minimal number of edges are depicted on fig. 10 a, b, c, e, f). For $n = 4k$ we have bipartite and non bipartite examples, for $n = 4k + 2$ we have bipartite example only, in other cases the graphs are non bipartite.



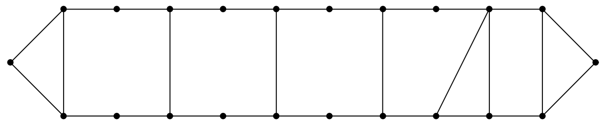
a) $n = 4k$, $r = 5k + 1$, bipartite graph



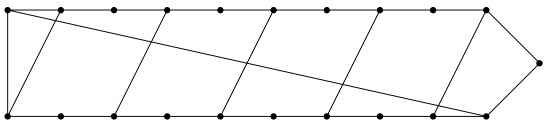
b) $n = 4k$, $r = 5k + 1$, non bipartite graph



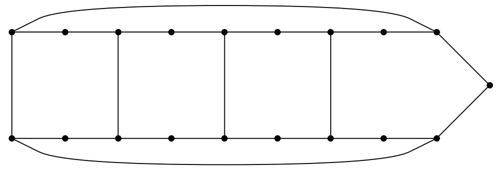
c) $n = 4k + 2$, $r = 5k + 3$, bipartite graph



d) $n = 4k + 2$, $r = 5k + 4$, non bipartite graph



e) $n = 4k + 1$, $r = 5k + 2$, non bipartite graph



f) $n = 4k + 3$, $r = 5k + 4$, non bipartite graph

Figure 10: Magic graphs with minimal number of vertices

The proof that the depicted graphs are magic consists of the routine verification that criterion from problems 3.2 holds. Instead of this this verification we show magic sets of weights for "typical case". Of course, this is not the proof but it follows that in concrete cases we really have magic graphs whose edges are separated in the spirit of the criterion. In general case the graphs will be magic too because the separation of its edges takes place "by the same reasons" as in this concrete examples. We restrict ourselves with case $n = 4k + 3$, $r = 5k + 4$, $k = 2$; see fig. 11.

3.11. This solution is a variation of [3]. Observe that if we add an arbitrary edge to non bipartite, connected (and non complete) magic graph then it remains be magic.

Indeed, the new edge e belongs to some cycle. If this cycle is even, assign the weight ε to this edge and the weight $\pm\varepsilon$ alternatively to all other edges of the cycle. Choose the value of ε so that all the weights remain positive and distinct. We obtain a magic set of weights.

If the cycle is odd choose another cycle that does not contain e (it exists because the graph is non bipartite). Then e belongs to some dum-bell (see lemma from solution 4.1) Once again, we can assign the weight ε to edge e and weights $\pm\varepsilon$, $\pm 2\varepsilon$ to other edges of dum-bell and obtain a magic set of weights.

Thus, to complete the solution it is sufficient for each $n \geqslant 5$ to construct "minimal" non-bipartite graph. It was done in the solution of the previous problem for $n \neq 4k + 2$ (see fig. 10 b, e, f). We found these examples in [3]. Unfortunately, the construction of minimal non-bipartite graph for $n = 4k+2$ in this article is wrong. In addition, the graph on fig. 10 b) is not magic for $n = 8$ (it is impossible to separate by 1-2-skeletons the slanted and the lowest edges). Magic non bipartite graph for $n = 8$ is depicted on fig. 9 d), it was invented by A. Tsybyshev. We do not know wether non bipartite magic graph with $4k + 2$ vertices and $5k + 3$ edges ($k > 3$) exists, so for the case of $5k + 3$ edges we leave bipartite example, and begin our construction from non bipartite graph with $5k + 4$ edges. This non bipartite graph with $4k + 2$ vertices and $5k + 4$ edges is depicted on fig. 10 d). Magic weights on this graph for $k = 3$ see on fig. 12.

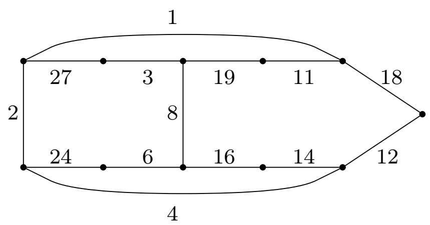


Figure 11: $n = 4k + 3$, $r = 5k + 4$, $k = 2$

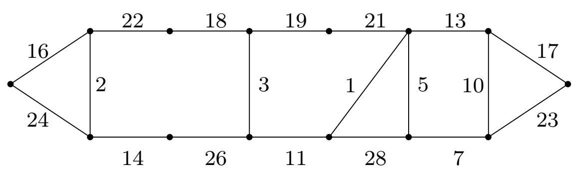


Figure 12: $n = 4k + 2$, $r = 5k + 4$, $k = 3$

4 Regular graphs

4.1. Solution 1.

Lemma. If a connected graph contains two odd cycles, and an edge e belongs to only one of them, than there exists an even cycle or a dum-bell that containns edge e.

P r o o f. The statement is trivial if cycles do not intersect or intersect by one vertex.

Assume that intersection of the cycles contains at least two vertices. Let X and Y be the endpoints of edge e. Remove edge e from the first cycle, the remaining part of this cycle we will call the path XY. Let A and B be the first and the last vertex in the path XY that belong to the second cycle. Then the parts XA and BY of the path XY do not intersect the second cycle. Vertices A and B split the second cycle onto two paths whose number of vertices have different parity. Adding one of them to paths XA and BY we can obtain an odd path $XABY$. Together with edge XY it forms an even cycle.

Let us return to the statement of the problem. W.l.o.g. we may assume that the graph is connected. Remove an arbitrary edge $e = AB$ from the graph. Assume that graph $G \setminus e$ falls to two components of connectivity. Since the degrees of all vertices were greater than 1 these components contain more than one vertex. Either of components can not be a bipartite graph. This is because in the bipartite graph the sums of degrees of all vertices in parts are equal, but in our components one of sums is divisible by d and another (which contains vertex A or B) is not divisible by d. Therefore each component contains an odd cycle. Hence edge e belongs to some dum-bell.

Now assume that the graph $G \setminus e$ is connected. Consider an arbitrary path from A to B. If this path has odd number of edges then e is contained in an even cycle. Assume that this path is even (then e is contained in odd cycle). If graph $G \setminus e$ is bipartite then both vertices A and B are in the same part. But this is impossible because the sums of degrees in the parts are not equal: first sum is equivalent $-2 \pmod d$ and the second sum is divisible by d. Hence graph $G \setminus e$ is non bipartite and there exists an odd cycle that does not contain e. It remains to use lemma.

S o l u t i o n 2 (by A. Tsybyshev). Assign weights $1/d$ to all edges of our regular graph. Then sum of weights in each vertex is equal to 1. Now start the process of destroying even cycles and dum-bells by changing weights and removing zero-weight edges that is described in the solution 2.3. As a result of this process we obtain a 1-2-skeleton with magic set of weights. Since we do not change the sum of weights in each vertex the sum of weights in each vertex is still equal to 1. Therefore the weights of edges are 1 and 1/2. Since $d \neq 1,\ 1/2,\ 0$ we change the weight of every edge at least once. Hence each edge belongs to some pseudocycle.

4.2. The placement of numbers ± 1 on even cycles and ± 1, ± 2 on dum-bells that was described in section 4 (problems) we will call *standard* weights on pseudocycle.

L e m m a. Let every edge of the graph has non zero (not necessarily positive) weight and sum of weights in each vertex is equal to 0. Then every edge is contained in even cycle or dum-bell.

The proof is analogous to the solution of the problem 4.1.

1. Assume that the regular graph G is magic and sum of weights in each vertex is equal to s. Subtract the number s/d from every weight. We obtain a placement of pairwise distinct numbers on edges of the graph with zero sum in each vertex.

Choose in the graph G an arbitrary edge of non zero weight and a pseudocycle that contains this edge (it exists due to lemma). Subtract from the weights of this pseudocycle the standard weights of the pseudocysle multiplyed by the appropriate coefficient in order to make the weight of chosen edge to be zero. Then remove all zero-weight edges. We obtain a placement with zero sum in each vertex. Then repeat this operation and so on.

In each step we decrease the number of edges therefore sooner or later all edges become zero-weight It means that the initial placement of numbers is a “sum” with appropriate coefficient of standard placements for pseudocycles. Since any two edges have distinct weights in the initial placement, there exists a pseudocycle such that its standard weights for this edges are distinct. By definition this pseudocycle weakly separates these edges.

2. Assume that every two edges are weakly separated by pseudocycles. Let us enumerate all pseudocycles. For k-th pseudocycle assign weights of its edges to be the standard placement multiplyed by 5^k. For each edge sum up its weights over all pseudocycles and after that add a large positive constant in order to make all weights positive. The obtained set of weights is magic.

4.3. It follows from the previuous problem.

4.4. 1. Let us check first that $\ell(G) \neq 1$. If $\ell(G) = 1$ then we can remove an edge e and obtain the graph with two components of connectivity. Each component itself is a magic graph, one of its vertices has degree $d-1$ and all others have degree d. This is impossible (see solution 4.1).

2. Prove that if $\ell(G) \geqslant 3$, then graph G is magic. Choose any two edges e and f and check that they are weakly separated by pseudocycles. After removing these edges the graph remains connected. Therefore there exists a cycle that contains e and does not contains f. Since the graph is bipartite this cycle is even and it separates edges e and f.

3. Prove that for $\ell(G) = 2$ graph G is non magic. If after removing edges e and f from graph G we obtain a disconnected graph then it has two components, say V and W, each of them is a bipartite graph. If the edges e and f have common endpoint then one of components has unique vertex of degree $d-2$ and other vertices of degree d. Such graph can not be bipartite. Therefore $e = AB$ and $f = CD$ have no common vertices, one component contains vertices A and C, another component contains vertices B and D of degree $d-1$. Hence B and D are in different parts of the component and all the paths that join these points have odd number of edges.

Now prove that edges e and f are not weakly separated by a pseudocycle. Since G is bipartite there are no even cycles (and dum-bells) in it. And all even cycles that contain both e and f do not separate these edges due to previous paragraph.

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