4

Colorings and clusters

M. Matdinov

Additional problems

23.* k-dimensional cube $n \times n \times \cdots \times n$ is divided by n^k k-dimensional subcubes $1 \times 1 \times \cdots \times 1$ colored by ℓ colors. Consider all triples of colors (a, b, c). Consider the set of points colored by these 3 colors simultaneously. Surround each of them by circle of radius 2. Consider connected components of the union of these circles. Suppose that all such components for all triples (a, b, c) have diameter less than d.

Then there exists a positive constant $C(k, d, \ell) > 0$ such that for any coloring there exists a cluster of volume $C(k, d, \ell) \cdot n^{k-1}$.

a) Prove that for $k = 3$.

b) Prove that for all k.

24.** Generalize condition of the previous problem for m-tuples of colours. General hypothesis: there exists a constant $C(k, m, d, \ell) > 0$ such that for each coloring of k-dimensional cube $n \times n \times \cdots \times n$ by l colors there exists a cluster of volume $C(k, m, d, \ell) \cdot n^{k+2-m}$.

This hypothesis can be considered as generalization of the problem 15. We don't know how to solve it.