Shapiro’s inequality

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1 Shapiro’s inequality

In October, 1954 the American Mathematical Monthly published the following problem of Harold Shapiro:

Prove the following inequality for positive numbers $x_1, x_2, \ldots, x_n$:

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \ldots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geq \frac{n}{2},$$

(1)

the equality holds only if all the denominators are equal.

In contrast to, say, “Kvant” magazine, it was allowed to publish problems in the Monthly, which were not solved by the proposer, and the readers had not been informed about this nuance. This time the situation was exactly like that. The author had a solution for partial cases $n = 3$ and 4 only.

In the following problems we can replace the condition that all the $x_k$’s are positive with the condition that all the $x_k$’s are nonnegative and all the denominators are nonzero. Indeed, if the inequality is proven for positive numbers, then it is not difficult to deduce the inequality for nonnegative numbers (and nonzero denominators). Let

$$f(x_1, x_2, \ldots, x_n) = \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \ldots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2}.$$ 

1.1. Prove the inequality (1) for $n = 3, 4, 5, 6$.

1.2. Prove that the inequality (1) is wrong

a) for $n = 20$; b) for $n = 14$; c) for $n = 25$.

1.3. Prove the inequality (1) for monotonic sequences.

1.4. Prove that if the inequality (1) does not hold for $n = m$, then it does not hold for $n = m + 2$.

1.5. Prove that if the inequality (1) does not hold for $n = m$, where $m$ is odd, then it does not hold for all $n > m$.

1.6. Prove the inequality (1) for $n = 8, 10, 12$ and for $n = 7, 9, 11, 13, 15, 17, 19, 21, 23$. Due to the statement of the previous problem it is sufficient to prove the inequality only for $n = 12$ and $n = 23$.

1.7. Prove that $f(x_1, x_2, \ldots, x_n) + f(x_n, x_{n-1}, \ldots, x_1) \geq n$.

1.8. Assume that the function $f(x_1, x_2, \ldots, x_n)$ has a local minimum in the point $(a_1, a_2, \ldots, a_n)$, $a_1, a_2, \ldots, a_n > 0$.

a) Prove that $f(a_1, a_2, \ldots, a_n) = n/2$ if $n$ is even.

b) Prove the same statement for odd $n$.

c) Use the statements a) and b) to prove the inequality for $n = 7$ and $n = 8$.

1.9. Prove the inequality $f(x_1, x_2, \ldots, x_n) \geq cn$ for the following values of the constant $c$:

a) $c = 1/4$; b) $c = (\sqrt{2} - 1)$; c) $c = 5/12$.

2 Useful and related inequalities

Prove the following inequalities assuming that all the $x_k$’s are positive. Prove that the constants printed in bold can not be decreased (for each $n$).

2.1. Mordell’s inequality.

a) $\left( \sum_{k=1}^{n} x_k \right)^2 \geq \min \left\{ \frac{n}{2}, 3 \right\} \sum_{k=1}^{n} x_k(x_{k+1} + x_{k+2})$.

b) Find all $n$-tuples $x_1, x_2, \ldots, x_n$ such that the equality is achieved.

2.2. $\left( \sum_{k=1}^{n} x_k \right)^2 \geq \min \left\{ \frac{n}{3}, 8 \right\} \sum_{k=1}^{n} x_k(x_{k+1} + x_{k+2} + x_{k+3})$.

2.3. mSa) Prove that for $n \leq 8$

$$\frac{x_1}{x_2 + x_3 + x_4} + \frac{x_2}{x_3 + x_4 + x_5} + \ldots + \frac{x_{n-1}}{x_n + x_1 + x_2} + \frac{x_n}{x_1 + x_2 + x_3} \geq \frac{n}{3},$$

b) For which $n > 8$ this inequality is also true?
2.5. \( \sum_{k=1}^{n} \frac{x_k}{x_{k+1} + x_{k+2}} \geq \sum_{k=1}^{n} \frac{x_{k+1}}{x_k + x_{k+1}}. \)

2.6. \( \frac{x_1}{x_n + x_2} + \frac{x_2}{x_1 + x_3} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_n} + \frac{x_n}{x_{n-1} + x_1} \geq 2; \quad n \geq 4. \)

2.7. \( \frac{x_1}{x_n + x_3} + \frac{x_2}{x_1 + x_4} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_1} + \frac{x_n}{x_{n-1} + x_2} \geq 3; \quad n \geq 4. \)

2.8. \( \frac{x_1}{x_n + x_3} + \frac{x_2}{x_1 + x_4} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_1} + \frac{x_n}{x_{n-1} + x_2} \geq 4; \quad n \geq 4. \)

2.9. \( \frac{x_1}{x_n + x_3} + \frac{x_2}{x_1 + x_4} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_1} + \frac{x_n}{x_{n-1} + x_2} \geq 6; \quad n \geq 6. \)

2.10. \( \frac{x_1}{x_n + x_3} + \frac{x_2}{x_1 + x_4} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_1} + \frac{x_n}{x_{n-1} + x_2} \geq 6. \)

2.11. \( \frac{x_1}{x_n + x_3} + \frac{x_2}{x_1 + x_4} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_1} + \frac{x_n}{x_{n-1} + x_2} \geq 4, \) where \( n > 5 \) is even.

2.12. \( \sum_{k=1}^{n} \frac{x_k^2}{x_{k+1}x_{k+2}} \geq \left( \frac{n+1}{2} \right). \)

3 After the intermediate finish

1.10. a) Prove that for each \( n \) there exists \( q_n > 1 \), such that for all real \( x_1, x_2, \ldots, x_n \in \left[ \frac{1}{q_n}; q_n \right] \) the inequality (1) holds.

b) Is it possible to choose \( q > 1 \), such that for all integers \( n > 0 \) and for all \( x_i \in \left[ \frac{1}{q}; q \right] \) the inequality (1) holds?

1.11. Let \( S = f(x_1, x_2, \ldots, x_n) \) be the left hand side of Shapiro’s inequality. Denote by \( a_1, a_2, \ldots, a_n \) the numbers \( x_2/x_1, x_3/x_2, \ldots, x_n/x_{n-1}, x_1/x_n \), arranged in increasing order.

a) Prove that \( S \geq \frac{1}{a_1(1+a_n)} + \frac{1}{a_2(1+a_{n-1})} + \ldots + \frac{1}{a_n(1+a_1)}; \)

b) Let \( b_k = \left\{ \begin{align*} \frac{a_k a_{n+1-k}}{a_k a_{n+1-k} + (a_k a_{n+1-k})^2} & \quad a_k a_{n+1-k} \geq 1 \\ \frac{a_k a_{n+1-k}}{a_k a_{n+1-k} + a_k a_{n+1-k}} & \quad a_k a_{n+1-k} < 1. \end{align*} \right. \) Prove that \( 2S \geq b_1 + b_2 + \ldots + b_n; \)

c) Let \( g \) be the maximal convex function that does not exceed both functions \( e^{-x} mS 2(e^x + e^{x/2})^{-1} \). Prove that \( 2S \geq g(\ln(a_1 a_n)) + g(\ln(a_2 a_{n-1})) + \ldots + g(\ln(a_n a_1)) \geq nq(0). \)

d) Prove that for each \( \lambda > g(0) \) there exist a nonnegative integer \( n \) and positive numbers \( x_1, x_2, \ldots, x_n \), such that \( S \leq \lambda n. \)
1.1. \( n = 3 \). Let \( S = x_1 + x_2 + x_3 \). It is easy to see that the function \( f(t) = \frac{t}{S-t} \) is convex on the interval \([0; S)\). Apply the Jensen inequality to it:

\[
\frac{f(x_1) + f(x_2) + f(x_3)}{3} \geq f\left(\frac{x_1 + x_2 + x_3}{3}\right) = f\left(\frac{S}{3}\right) = \frac{1}{2}.
\]

We are done.

\( n = 4 \). This inequality is cyclic. Write down the values of \( x_i \)'s successively at the vertices of a square. Then on each diagonal put an arrow leading from the smaller value to the greater one. Notice that there is a side of the square with two tails on it. Re-number the \( x_i \)'s in such a manner that this side becomes \( x_1 x_1 \). Now we may assume that \( x_1 \geq x_3, x_4 \geq x_2 \). For the variables with these restrictions the following inequality is true:

\[
\frac{x_1}{x_2 + x_3} + \frac{x_3}{x_4 + x_1} \geq \frac{x_1}{x_4 + x_3} + \frac{x_3}{x_2 + x_1}.
\]

Indeed, re-write it in the following way:

\[
x_1\left(\frac{1}{x_2 + x_3} - \frac{1}{x_4 + x_3}\right) \geq x_3\left(\frac{1}{x_2 + x_1} - \frac{1}{x_4 + x_1}\right).
\]

Reduce both hands to a common denominator, cancel \( x_1 - x_2 \) in both hands (if \( x_1 - x_2 = 0 \), we already have the equality), and multiply both hands to the product of denominators. We obtain the evident (since \( x_1 \geq x_3 \)) inequality

\[
x_1(x_2 + x_1)(x_4 + x_1) \geq x_3(x_2 + x_3)(x_4 + x_3).
\]

Use it to prove Shapiro’s inequality:

\[
\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \frac{x_3}{x_4 + x_1} + \frac{x_4}{x_1 + x_2} \geq \frac{x_1}{x_4 + x_3} + \frac{x_2}{x_3 + x_4} + \frac{x_3}{x_2 + x_1} + \frac{x_4}{x_1 + x_2} = \frac{x_1 + x_2}{x_3 + x_4} + \frac{x_3 + x_4}{x_1 + x_2} = a + a^{-1} \geq 2.
\]

\( n = 5 \). Notice that the function \( f(t) = 1/(S - t) \) is convex on the interval \([0; S)\). So we can apply the Jensen inequality with \( n = 5 \):

\[
a_1 f(t_1) + a_2 f(t_2) + a_3 f(t_3) + a_4 f(t_4) + a_5 f(t_5) \geq f(a_1 t_1 + a_2 t_2 + a_3 t_3 + a_4 t_4 + a_5 t_5),
\]

where \( a_i \geq 0, \sum a_i = 1 \). Take \( a_i = \frac{x_i}{S} \), and let \( t_i = x_i + x_{i-1} + x_{i-2}, i = 1, \ldots, 5 \) (we assume that the variables are enumerated cyclically: \( x_0 = x_5, x_1 = x_0 \)). Then \( f(t_i) = \frac{1}{S - t_i} = \frac{S}{x_i + x_{i-1} + x_{i-2}}, \) and it means that the left-hand side of inequality (2) coincides with the left-hand side of Shapiro’s inequality. Now consider the right-hand side of 2:

\[
\frac{1}{S - \sum_{i=1}^{n} a_i t_i} = \frac{1}{S - \sum_{i=1}^{n} \frac{x_i}{S}(x_i + x_{i-1} + x_{i-2})} = \frac{S}{S - \sum_{i=1}^{n} x_i(x_i + x_{i-1} + x_{i-2})}.
\]

Open the brackets. It is easy to see that the denominator is the sum of pairwise products of the set of variables \( x_i \). Since the initial inequality is homogeneous, we may assume that \( S = x_1 + x_2 + x_3 + x_4 + x_5 = 1 \). Now the right-hand side of inequality (2) is the inverse number to the sum of pairwise products of the variables \( x_i \), satisfying one condition \( x_1 + x_2 + x_3 + x_4 + x_5 = 1 \). The right-hand side reaches its minimum when the sum of pairwise products reaches its maximum. It is well-known that for it all the variables should be equal. But the right-hand side equals \( 5/2 \) in this point.

The analogous proof also works for \( n = 4 \).

\( n = 6 \). Proceed as above. The function \( f(t) = 1/(S - t) \) is convex on the interval \([0; S)\). So we can apply the Jensen inequality with \( n = 6 \):

\[
\sum_{i=1}^{n} a_i f(t_i) \geq f\left(\sum_{i=1}^{n} a_i t_i\right).
\]

Let \( a_i = \frac{x_i}{S}, t_i = x_i + x_{i-1} + x_{i-2} + x_{i-3}, i = 1, \ldots, 6 \) (we assume that the variables are enumerated cyclically: \( x_0 = x_6, x_1 = x_5, x_2 = x_4 \)). Then \( f(t_i) = \frac{1}{S - t_i} = \frac{S}{x_i + x_{i-1} + x_{i-2}}, \) and this means that the left-hand side of the inequality (1.1) coincides with the left-hand side of Shapiro’s inequality. Now consider the right-hand side of (1.1):

\[
\frac{1}{S - \sum_{i=1}^{n} a_i t_i} = \frac{1}{S - \sum_{i=1}^{n} \frac{x_i}{S}(x_i + x_{i-1} + x_{i-2} + x_{i-3})} = \frac{S}{S - \sum_{i=1}^{n} x_i(x_i + x_{i-1} + x_{i-2} + x_{i-3})}.
\]
Open the brackets. It is easy to see that the denominator is the sum of pairwise products of the variables $x_i$'s but the products $x_1 x_4$, $x_2 x_5$, and $x_3 x_6$. This sum can be re-written as $(x_1 + x_4)(x_2 + x_5) + (x_1 + x_4)(x_3 + x_6) + (x_2 + x_5)(x_3 + x_6)$. Denote $A = x_1 + x_4$, $B = x_2 + x_5$, $C = x_3 + x_6$. The right-hand side of (1.1) can be re-written as

$$\frac{A + B + C}{AB + BC + AC}.$$  

(3)

Since the initial inequality is homogeneous, we may assume that $S = x_1 + x_2 + x_3 + x_4 + x_5 = A + B + C = 1$. Now it is clear that the expression (3) is greater than or equal to 3, since $(A + B + C)^2 \geq 3(AB + BC + AC)$. 

**Remark.** Unfortunately, this method does not work for $n > 6$.

**Second solution.** Apply the Cauchy-Bunyakovsky inequality to the sets of numbers

$$\sqrt{\frac{x_1}{x_2 + x_3}}, \sqrt{\frac{x_2}{x_3 + x_4}}, \ldots, \sqrt{\frac{x_n}{x_1 + x_2}} \quad \text{and} \quad \sqrt{x_1(x_2 + x_3)}, \sqrt{x_2(x_3 + x_4)}, \ldots, \sqrt{x_n(x_1 + x_2)}.$$ 

We obtain

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \ldots + \frac{x_n}{x_1 + x_2} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{x_1(x_2 + x_3) + x_2(x_3 + x_4) + \ldots + x_n(x_1 + x_2)}.$$ 

Use Mordell's inequality (problem 2.1). When $n \leq 6$, it gives us that the right-hand side of this inequality is greater than or equal to $n/2$.

**1.2. a)** [22] Take as $x_1$, $x_2$, ..., $x_{20}$ numbers

$$1 + 5\varepsilon, \quad 6\varepsilon, \quad 1 + 4\varepsilon, \quad 5\varepsilon, \quad 1 + 3\varepsilon, \quad 4\varepsilon, \quad 1 + 2\varepsilon, \quad 3\varepsilon, \quad 1 + \varepsilon, \quad 2\varepsilon, \quad 1 + 2\varepsilon, \quad \varepsilon, \quad 1 + 3\varepsilon, \quad 2\varepsilon, \quad 1 + 4\varepsilon, \quad 3\varepsilon, \quad 1 + 5\varepsilon, \quad 4\varepsilon, \quad 1 + 6\varepsilon, \quad 5\varepsilon.$$ 

Then $f(x_1, \ldots, x_{20}) < 10 - \varepsilon^2 + c\varepsilon^3 < 10$ for some $c$ and small enough $\varepsilon$.

**b)** [27] Take as $x_1$, $x_2$, ..., $x_{14}$ numbers

$$1 + 7\varepsilon, \quad 7\varepsilon, \quad 1 + 4\varepsilon, \quad 6\varepsilon, \quad 1 + \varepsilon, \quad 5\varepsilon, \quad 1, \quad 2\varepsilon, \quad 1 + \varepsilon, \quad 0, \quad 1 + 4\varepsilon, \quad \varepsilon, \quad 1 + 6\varepsilon, \quad 4\varepsilon.$$ 

Then $f(x_1, \ldots, x_{20}) < 7 - 2\varepsilon^2 + c\varepsilon^3 < 7$ for some $c$ and small enough $\varepsilon$.

**An alternative example [24]:**

$$0, \quad 42, \quad 2, \quad 42, \quad 4, \quad 41, \quad 5, \quad 39, \quad 4, \quad 38, \quad 2, \quad 38, \quad 0, \quad 40.$$ 

c) [10], [18]. Take

$$0, \quad 85, \quad 0, \quad 101, \quad 0, \quad 120, \quad 14, \quad 129, \quad 41, \quad 116, \quad 59, \quad 93, \quad 64, \quad 71, \quad 63, \quad 52, \quad 60, \quad 36, \quad 58, \quad 23, \quad 58, \quad 12, \quad 62, \quad 3, \quad 71.$$ 

Alternatively, in [3] the following example is given:

$$32, \quad 0, \quad 37, \quad 0, \quad 43, \quad 0, \quad 50, \quad 0, \quad 59, \quad 8, \quad 62, \quad 21, \quad 55, \quad 29, \quad 44, \quad 32, \quad 33, \quad 31, \quad 24, \quad 30, \quad 16, \quad 29, \quad 10, \quad 29, \quad 4.$$ 

**1.3.** The statement of the problem is published in [13]. We present here a short nice solution. Let $x_1 \geq x_2 \geq \ldots \geq x_n > 0$. Observe that the product of $n$ fractions $\frac{x_{k+1} + x_{k+2}}{x_{k+1} + x_{k+2}}$ is equal to 1. Then by Cauchy inequality we conclude that

$$\sum_{k=1}^{n} \frac{x_k}{x_{k+1} + x_{k+2}} \geq n = \sum_{k=1}^{n} \frac{x_{k+1} + x_{k+2}}{x_{k+1} + x_{k+2}}.$$ 

Hence

$$\sum_{k=1}^{n} \frac{x_k}{x_{k+1} + x_{k+2}} \geq \sum_{k=1}^{n} \frac{x_{k+2}}{x_{k+1} + x_{k+2}} = \sum_{k=1}^{n} \frac{x_{k+1}}{x_{k+1} + x_{k+2}}.$$  

(4)

Now we will apply the rearranging inequality: Let $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ be two sets of numbers. Then for each permutation $k_1, \ldots, k_n$ of numbers 1, ..., $n$ the following inequality holds

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \geq a_1 b_{k_1} + a_2 b_{k_2} + \ldots + a_n b_{k_n} = a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1.$$
Use the rearranging inequality twice
\[
\sum_{k=1}^{n} \frac{x_k}{x_k+1 + x_{k+2}} = \frac{n-2}{x_{k+1} + x_{k+2}} + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} + \frac{x_n}{x_1 + x_2} > 0
\]
\[
\sum_{k=1}^{n} \frac{x_k}{x_k+1 + x_{k+2}} + \frac{x_{n-1}}{x_1 + x_2} + \frac{x_n}{x_1 + x_2} \geq 1
\]
\[
\sum_{k=1}^{n} \frac{x_k}{x_k+1 + x_{k+1}} + \frac{x_{n+1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geq 1
\]
The inequality (*) here is the rearranging inequality for two pairs of numbers: \(x_{n-1} > x_n\) and \(\frac{1}{x_n+x_1} > \frac{1}{x_1+x_2}\); and the inequality (**) is the rearranging inequality for the sets \(x_1, x_2, \ldots, x_{n-1}\) and \(\frac{1}{x_1+x_2}, \frac{1}{x_2+x_3}, \ldots, \frac{1}{x_{n-1}+x_n}\) that have opposite ordering.

Thus
\[
2 \sum_{k=1}^{n} \frac{x_k}{x_k+1 + x_{k+2}} \geq \sum_{k=1}^{n} \frac{x_k}{x_k+1 + x_{k+1}} + \sum_{k=1}^{n} \frac{x_{n+1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} = n.
\]

For the decreasing set \(x_i\) the solution is similar because we do not use the order of the variables when we apply the Cauchy inequality, and for the rearranging inequalities we need the fact that the sets \(x_i\) and \(\frac{1}{x_i+x_{i+1}}\) have different orderings.

1.4. [3] It is easy to see that \(f_{n+2}(x_1, x_2, \ldots, x_n, x_1, x_2) = f_n(x_1, x_2, \ldots, x_n) + 1\). Therefore if \(f_n(x_1, x_2, \ldots, x_n) < n/2\), then \(f_{n+2}(x_1, x_2, \ldots, x_n, x_1, x_2) < (n + 2)/2\).

1.5. [3] Assume that \(f_m(x_1, x_2, \ldots, x_m) < \frac{m}{2} - 1\). For each \(k\) let us calculate the difference
\[
f_{m+1}(x_1, \ldots, x_k, x_k, x_{k+1}, \ldots, x_m) - f_m(x_1, x_2, \ldots, x_m) = \frac{x_{k-1}}{2}x_k + \frac{x_k}{x_k + x_{k+1}} - \frac{x_{k-1}}{x_k + x_{k+1}} - \frac{1}{2} = (x_k - x_{k-1})(x_k - x_{k+1})
\]

If \((x_k - x_{k-1})(x_k - x_{k+1}) \leq 0\), then
\[
f_{n+1}(x_1, x_2, \ldots, x_k, x_k, x_{k+1}, \ldots, x_m) \leq \frac{m}{2} - 1
\]
and we are done. If \(n\) is odd, we can always choose \(k\) such that \((x_k - x_{k-1})(x_k - x_{k+1}) \leq 0\) because otherwise the product of the (odd number of) inequalities \((x_k - x_{k-1})(x_{k+1} - x_k) < 0\) for all \(k\) is \((x_2 - x_1)^2(x_3 - x_2)^2 \ldots (x_m - x_{m-1})^2(x_1 - x_m)^2 < 0\).

Thus if for odd \(n\) the Shapiro inequality is wrong then for \(n + 1\) it is wrong, too. It remains to apply the statement of the previous problem.

1.6. [7, 8]

1.7. [28] Let \(y_k = x_k + x_{k+1}\). Then
\[
\frac{x_1 + x_4}{x_2 + x_3} + \frac{x_2 + x_5}{x_3 + x_4} + \ldots + \frac{x_{n+3}}{x_{n+2}} = \sum_{k=1}^{n} \frac{y_k}{y_k+1} = \sum_{k=1}^{n} \frac{y_k}{y_k+1} + \sum_{k=1}^{n} \frac{y_k+1}{y_k+1} - n \geq n,
\]
because by Cauchy inequality each sum is at least \(n\).

1.8. The statements a), b) were published in [21].

a) !!! This short proof is taken from [8].

Denote for brevity \(a = (a_1, a_2, \ldots, a_n)\), \(x = (x_1, x_2, \ldots, x_n)\), and \(u = (-1, 1, -1, 1, \ldots, -1, 1)\).

Observe that
\[
\frac{\partial f}{\partial x_k}(x) = \frac{1}{x_{k+1} + x_{k+2}} - \frac{x_{k-2}}{(x_{k-1} + x_k)^2} - \frac{x_{k-1}}{(x_k + x_{k+1})^2}.
\]

It is easy to see that we have an identity
\[
f(x + tu) = f(x) + t \sum_{k=1}^{n} (-1)^k \frac{\partial f}{\partial x_k}(x).
\]
Since \(a\) is the minimum point, we have
\[
\frac{\partial f}{\partial x_k}(a) = 0.
\]
Therefore \( f(a + tu) = f(a) \) if all the coordinates of the point \( a + tu \) are positive. Hence \( a + tu \) is the minimum point of the function \( f \) as well. Hence,

\[
\frac{\partial f}{\partial x_k}(a + tu) = 0.
\]

So

\[
\frac{1}{a_{k+1} + a_{k+2}} - \frac{a_{k-2}}{(a_{k-1} + a_k)^2} - \frac{a_{k-1}}{(a_k + a_{k+1})^2} = 0
\]

and

\[
\frac{1}{a_{k+1} + a_{k+2}} - \frac{a_{k-2} + t(-1)^{k-2}}{(a_{k-1} + a_k)^2} - \frac{a_{k-1} + t(-1)^{k-1}}{(a_k + a_{k+1})^2} = 0.
\]

Subtract the first equality from the second:

\[
\frac{t}{(a_{k-1} + a_k)^2} - \frac{t}{(a_k + a_{k+1})^2} = 0.
\]

Therefore,

\[
a_{k-1} + a_k = a_k + a_{k+1}.
\]

and hence

\[
a_1 = a_3 = a_5 = \cdots = a_{n-1} \quad \text{and} \quad a_2 = a_4 = a_6 = \cdots = a_n.
\]

Thus, \( f(a) = n/2 \).

b) This short proof is taken from [7]. Denote for brevity \( a = (a_1, a_2, \ldots, a_n) \), \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \), \( z = (z_1, z_2, \ldots, z_n) \), where \( y_k = x_k + x_{k+1} \) and \( z_k = 1/y_{n+1-k} \).

Set

\[
S(x) = \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} = \sum_{k=0}^{n} \frac{x_k}{y_{k+1}}.
\]

Observe that

\[
\frac{\partial f}{\partial x_k}(x) = \frac{1}{x_{k+1} + x_{k+2}} - \frac{x_{k-2}}{(x_{k-1} + x_k)^2} - \frac{x_{k-1}}{(x_k + x_{k+1})^2}.
\]

It is easy to check the following identities:

\[
\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d} = \frac{\frac{a}{b} + \frac{c}{d}}{1 + \frac{c}{d}}.
\]

Hence,

\[
\frac{x_{k-2} - x_{k-1}}{x_{k-1} + x_k} + \frac{x_{k-1} - x_k}{x_k + x_{k+1}} = \frac{x_{k-2} + x_{k-1}}{(x_{k-1} + x_k) + (x_k + x_{k+1})} + \frac{x_{k-2} - x_{k-1}}{(x_{k-1} + x_k) + (x_k + x_{k+1})} = \frac{1}{x_{k-1} + x_k} + \frac{1}{x_k + x_{k+1}} = \frac{y_{k-2}}{y_{k-1} + y_k} + \frac{z_{n-k}}{z_{n-k+1} + z_{n-k+2}}.
\]

Therefore,

\[
2S(x) = S(y) + S(z) - \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x) - \frac{\partial f}{\partial x_k}(x).
\]

If \( x \) is a minimum point then we have \( 2S(x) = S(y) + S(z) \). Hence \( S(x) = S(y) = S(z) \).

Let \( u := (x_1 + x_2 + \cdots + x_n)/n \). Consider the transformation \( M : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
M(x) = \left( \frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \ldots, \frac{x_n + x_1}{2} \right).
\]

Let \( M_k(x) \) be its \( k \)-th iteration. Observe that \( S(x) = S(y) = S(M(x)) = \cdots = S(M_k(x)) \). It is clear that \( \lim_{k \to \infty} M_k(x) = (u, u, \ldots, u) \). Then

\[
S(x) = \lim_{k \to \infty} S(M_k(x)) = S((u, u, \ldots, u)) = \frac{n}{2}.
\]
1.9. These solutions are taken from [3].

nS) The problem was presented at the Third USSR mathematical olympiad, 1969. Probably it was originally published in [14].

Let \( x_i \), be the maximal number among \( x_1, x_2, \ldots, x_n \); \( x_i \) be the maximum of the two next numbers after \( x_i \) (i.e. of \( x_{i+1} \) and \( x_{i+2} \)); \( x_i \) be the maximum of the two next numbers after \( x_i \), and so on. We will continue this sequence till the step number \( k \) when the maximum of the two next \( x_{ik} \) numbers is \( x_i \).

It is clear that \( k \geq n/2 \). We have

\[
\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_n}{x_1 + x_2} \geq \frac{x_i}{2x_i} + \frac{x_{i+1}}{2x_i} + \cdots + \frac{x_k}{2x_i}.
\]

The last expression is at least \( k/2 \) by the Cauchy inequality therefore it is at least \( n/4 \).

b) Rewrite each of the fractions \( \frac{x_k}{x_{k+1} + x_{k+2}}, k = 1, 2, \ldots, n \), in the form

\[
\frac{x_k}{x_{k+1} + x_{k+2}} = \frac{x_k + \frac{1}{2}x_{k+1}}{x_{k+1} + x_{k+2}} + \frac{1}{2}\frac{x_{k+1} + x_{k+2}}{x_{k+1} + x_{k+2}} - 1.
\]

We obtain \( 2n \) fractions. Combine them by pairs: the first and the last, the second and the third, the fourth and the fifth and so on. Now estimate the sum of each pair from below

\[
\frac{1}{2}x_k + x_{k+1} + x_k + \frac{1}{2}x_{k+1} + \frac{1}{2}x_{k+1} + x_{k+2} > 2\sqrt{\frac{1}{2}x_k + x_{k+1}}\frac{x_k + x_{k+1}}{x_{k+1} + x_{k+2}} - 1.
\]

Since the product \( n \) numbers \( \sqrt{x_1 + x_2}, \sqrt{x_2 + x_3}, \ldots, \sqrt{x_1 + x_2} \) equals 1, then by the Cauchy inequality their sum is at least \( n \). Therefore \( f(x_1, \ldots, x_n) \geq \sqrt{2n} - n = (\sqrt{2} - 1)n \).

c) As in the previous solution rewrite each of the fractions \( \frac{x_k}{x_{k+1} + x_{k+2}}, k = 1, 2, \ldots, n \), in the form

\[
\frac{x_k}{x_{k+1} + x_{k+2}} = \frac{x_k + \beta x_{k+1}}{x_{k+1} + x_{k+2}} + \alpha \cdot \beta x_{k+1} + x_{k+2} - x_{k+2} - \alpha,
\]

where \( \alpha \) and \( \beta \) are parameters chosen to make the equality true. For such a choice of \( \alpha \) and \( \beta \) we need \( \beta + \alpha \beta = \alpha \), i.e. \( \beta = \alpha/(\alpha + 1) \). Then

\[
\frac{x_k + \beta x_{k+1}}{x_{k+1} + x_{k+2}} + \alpha \cdot \beta x_{k+1} + x_{k+2} - x_{k+2} - \alpha > 2\sqrt{\alpha x_k + x_{k+1}}\frac{x_k + x_{k+1}}{x_{k+1} + x_{k+2}} > 2\sqrt{\frac{\alpha}{\sqrt{\alpha + 1}}}\frac{x_k + x_{k+1}}{x_{k+1} + x_{k+2}} > \frac{2\alpha}{\sqrt{\alpha + 1}}\cdot \sqrt{\frac{x_k + x_{k+1}}{x_{k+1} + x_{k+2}}}.
\]

Therefore

\[
\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geq \frac{2\alpha}{\sqrt{\alpha + 1}}\left(\sqrt{\frac{x_1 + x_2}{x_2 + x_3}} + \sqrt{\frac{x_2 + x_3}{x_3 + x_4}} + \cdots + \sqrt{\frac{x_n + x_1}{x_1 + x_2}} - \alpha n\right)
\]

\[
> \frac{2\alpha}{\sqrt{\alpha + 1}}n - \alpha n = \left(\frac{2\alpha}{\sqrt{\alpha + 1}} - \alpha \right)n.
\]

For \( \alpha = \frac{5}{4} \) we have \( c = 5/12 \).

Remark. This is a good approximation. The expression \( g(\alpha) = \frac{2\alpha}{\sqrt{\alpha + 1}} - \alpha \) reaches its maximal value at \( \alpha = \alpha_0 \approx 1.1479 \) (this is a root of the cubic equation \( g'(\alpha) = 0 \), and the minimum value is \( g(\alpha_0) \approx 0.4186 \). For \( \alpha = \frac{5}{4} \) we have \( g(\alpha) = \frac{5}{12} \approx 0.416 \).

1.10. [9]. Set \( y_k = x_k + x_{k+1} \). We need to prove that

\[
\frac{x_1}{y_2} + \frac{x_2}{y_3} + \cdots + \frac{x_n}{y_1} \geq \frac{n}{2},
\]
or
\[
\sum_{k=1}^{n} \frac{2q_n^2 x_k - y_{k+1}}{y_{k+1}} \geq n(q_n^2 - 1).
\]

We suppose that the parameter \( q_n \) will be chosen later. Since
\[
2q_n^2 x_k - y_{k+1} = (q_n^2 x_k - x_{k+1}) + (q_n^2 x_k - x_{k+2}) \geq 0,
\]
b by the Cauchy-Bunyakovsky inequality for sets
\[
\left\{ \sqrt{\frac{2q_n^2 x_k - y_{k+1}}{y_{k+1}}} \right\} \quad \text{and} \quad \left\{ \sqrt{(2q_n^2 x_k - y_{k+1})y_{k+1}} \right\}
\]
we have
\[
\sum_{k=1}^{n} \frac{2q_n^2 x_k - y_{k+1}}{y_{k+1}} \geq \frac{\left( \sum_{k=1}^{n} (2q_n^2 x_k - y_{k+1}) \right)^2}{\sum_{k=1}^{n} (2q_n^2 x_k - y_{k+1})y_{k+1}}.
\]

So it suffices to prove that
\[
A^2 := \left( \sum_{k=1}^{n} (2q_n^2 x_k - y_{k+1}) \right)^2 \geq n(q_n^2 - 1)\sum_{k=1}^{n} (2q_n^2 x_k - y_{k+1})y_{k+1} =: n(q_n^2 - 1)B.
\]

Since \( \sum_{k=1}^{n} y_k = 2 \sum_{k=1}^{n} x_k \), we have
\[
A = (q_n^2 - 1)\sum_{k=1}^{n} y_k,
\]
\[
B = 2q_n^2 \sum_{k=1}^{n} x_k y_{k+1} - \sum_{k=1}^{n} y_k^2 = 2q_n^2 \sum_{k=1}^{n} y_k y_{k+1} - (q_n^2 + 1)\sum_{k=1}^{n} y_k^2.
\]

So it remains to prove that
\[
(q_n^2 - 1)\left( \sum_{k=1}^{n} y_k \right)^2 \geq n \left( 2q_n^2 \sum_{k=1}^{n} y_k y_{k+1} - (q_n^2 + 1)\sum_{k=1}^{n} y_k^2 \right).
\]

Transform the left-hand side using the relation
\[
\left( \sum_{k=1}^{n} y_k \right)^2 = n \sum_{k=1}^{n} y_k^2 - \sum_{i<k} (y_i - y_k)^2.
\]
The inequality (5) will be transformed to
\[
n \sum_{k=1}^{n} (y_k - y_{k+1})^2 \geq \left( 1 - \frac{1}{q_n^2} \right) i<k (y_i - y_k)^2.
\]

By the Cauchy-Bunyakovsky inequality
\[
\sum_{k=1}^{n} (y_k - y_{k+1})^2 \geq \sum_{j=i}^{k-1} (y_j - y_{j+1})^2 \geq \frac{1}{k-j} \left( \sum_{j=i}^{k-1} (y_j - y_{j+1}) \right)^2 = \frac{1}{k-j} (y_i - y_k)^2 \geq \frac{1}{n-1} (y_i - y_k)^2.
\]

Hence
\[
n(n-1) \sum_{k=1}^{n} (y_k - y_{k+1})^2 \geq \frac{1}{n-1} \sum_{i<k} (y_i - y_k)^2.
\]

So we can take \( 1 - \frac{1}{q_n^2} = \frac{2}{n-1} \), i.e. \( q_n = \frac{n-1}{\sqrt{n-1} - 1} > 1 \).

Remark. When \( n \) tends to infinity, the values \( q_n \) which are found above tend to 1.

b) 1.11. (a) Denote \( k_i := x_{i+1}/x_i \). Then
\[
S = \frac{1}{k_1(k_2 + 1)} + \frac{1}{k_2(k_3 + 1)} + \cdots + \frac{1}{k_n(k_1 + 1)} \geq \frac{1}{a_1(a_n + 1)} + \frac{1}{a_2(a_{n-1} + 1)} + \cdots + \frac{1}{a_n(a_1 + 1)}.
\]
(b) The inequality holds because
\[
\frac{1}{a_i(a_{n+1-i}+1)} + \frac{1}{a_{n+1-i}(a_i+1)} = \frac{1 + a_i a_{n+1-i-1}}{a_i a_{n+1-i}} \geq b_i
\]
where the latter inequality holds because \((1 + a_i)(1 + a_{n+1-i}) \geq (1 + \sqrt{a_i a_{n+1-i}})^2\).

(c) The first inequality \(2S \geq g(\ln(a_1a_n)) + g(\ln(a_2a_{n-1})) + \ldots + g(\ln(a_na_1))\) holds because \(g(x)\) is less than both \(e^{-x}\) and \(2(e^x + e^{-x/2})^{-1}\). The second inequality holds by the Jensen inequality because \(g\) is convex.

(d) [Dr]

2.1. a) [20]

For \(n = 4\) we need to prove that
\[
(x_1 + x_2 + x_3 + x_4)^2 \geq 2x_1x_2 + 2x_2x_3 + 2x_3x_4 + 2x_4x_1 + 4x_1x_3 + 4x_2x_4.
\]
This follows from the inequality
\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq 2x_1x_3 + 2x_2x_4.
\]

For \(n = 3\) and \(n = 5\) re-write the inequality. We need to prove that
\[
(n - 1)(a_1 + a_2 + \ldots + a_n)^2 \geq 2n \sum_{i<k} a_ia_k. \tag{6}
\]
Indeed, notice that the Cauchy–Bunyakovsky inequality applied to sets \(a_1, a_2, \ldots, a_n\) and \(1, 1, \ldots, 1\) gives us:
\[
n(a_1^2 + a_2^2 + \ldots + a_n^2) \geq (a_1 + a_2 + \ldots + a_n)^2.
\]
Now we have
\[
n(a_1 + a_2 + \ldots + a_n)^2 = n(a_1^2 + a_2^2 + \ldots + a_n^2) + 2n \sum_{i<k} a_ia_k \geq (a_1 + a_2 + \ldots + a_n)^2 + 2n \sum_{i<k} a_ia_k,
\]
which implies (6).

Now assume that \(n \geq 6\). We may suppose that \(x_3 \geq x_1\) and \(x_3 \geq x_2\) (e.g. make a cyclic shift of variables such that \(x_3\) becomes the maximum). For \(r = 1, 2, 3\) denote by \(a_r\) the sum of all \(x_k\) such that \(k \equiv r \pmod{3}\) and \(k \leq n\). Then \(x_1 + x_2 + \ldots + x_n = a_1 + a_2 + a_3\). Hence by (6) we have
\[
(x_1 + x_2 + \ldots + x_n)^2 = (a_1 + a_2 + a_3)^2 \geq 3(a_1a_2 + a_2a_3 + a_3a_1) = 3 \cdot \sum_{(i-k),3} x_i x_k.
\]

Set
\[
A := \sum_{(i-k),3} x_i x_k \quad \text{and} \quad B := \sum_{k=1}^{n} x_k(x_{k+1} + x_{k+2}).
\]
We have \(A \geq B\) because

\begin{itemize}
  \item for \(n \equiv 0 \pmod{3}\) all the summands of \(B\) are contained in \(A\);
  \item for \(n \equiv 1 \pmod{3}\) the sum \(A\) contains all the summands of \(B\) except \(x_nx_1\), but \(x_nx_1\) does not exceed \(x_nx_3\);
  \item for \(n \equiv 2 \pmod{3}\) the sum \(A\) contains all the summands of \(B\) except \(x_{n-1}x_1\) and \(x_nx_2\), but these summands do not exceed \(x_{n-1}x_3\) and \(x_nx_3\), respectively.
\end{itemize}

Hence
\[
(x_1 + x_2 + \ldots + x_n)^2 \geq 3A \geq 3B = 3 \sum_{k=1}^{n} x_k(x_{k+1} + x_{k+2}).
\]

In order to show that \(\min\{\frac{\pi}{4}, 3\}\) is the sharp constant for \(n \leq 6\) we set \(x_1 = x_2 = \ldots = x_n = 1\) and for \(n \geq 6\) we set \(x_1 = x_2 = x_3 = 1\) and \(x_1 = x_5 = \ldots = x_n = 0\).

b) The case \(n < 6\) is trivial. For \(n = 6\) the equality is achieved when \(x_1 + x_4 = x_2 + x_5 = x_3 + x_6\). For \(n \geq 6\) the equality is achieved for the sets of form \((t, 1, 1, 1-t, 0, \ldots, 0)\), where \(t \in [0, 1]\), and their cyclic shifts.

2.2. [20]

For \(n = 4\) and \(n = 7\) this is a particular case of (6).

For \(n = 5\) the inequality coincides with \(\sum(x_k - 2x_{k+2} + x_{k+4})^2 \geq 0\).

For \(n = 6\) the inequality follows from \(x_1^2 + x_2^2 + \ldots + x_6^2 \geq 2x_1x_4 + 2x_2x_5 + 2x_3x_6\).

For \(n = 8\) open brackets in the following corollary of the Cauchy–Bunyakovsky inequality
\[
4(x_1^2 + x_2^2 + x_3^2 + x_4^2) \geq (x_1 + x_2 + x_3 + x_4)^2.
\]
We obtain
\[
3(x_1^2 + x_2^2 + x_3^2 + x_4^2) \geq 2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4).
\]
2.4. \[ (x_1 + x_2 + x_3 + x_4)^2 \geq 8(x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4), \] (7)

This is the required inequality for \( n = 8 \).

Now assume that \( n > 8 \). We may suppose that \( x_4 \geq x_1, x_4 \geq x_2, \) and \( x_4 \geq x_3 \). For \( r = 1, 2, 3, \) and 4 denote by \( a_r \) the sum of all \( x_k \) such that \( k \equiv r \pmod{4} \) and \( k \leq n \). Then \( x_1 + x_2 + \ldots + x_n = a_1 + a_2 + a_3 + a_4 \). Hence by \( (7) \)

\[ 3(x_1 + x_2 + \ldots + x_n)^2 = 3(a_1 + a_2 + a_3 + a_4)^2 \geq 8(a_1 a_2 + a_2 a_3 + a_3 a_4 + a_4 a_1) \geq 8 \cdot \sum_{(i-k)\neq 4} x_i x_k. \]

Set

\[ A := \sum_{(i-k)\neq 4} x_i x_k \quad \text{and} \quad B := \sum_{k=1}^{n} x_k(x_{k+1} + x_{k+2} + x_{k+3}). \]

We have \( A \geq B \) because

- for \( n \equiv 0 \pmod{4} \) all the summands of \( B \) are contained in \( A \);
- for \( n \equiv 1 \pmod{4} \) the sum \( A \) contains all the summands of \( B \) except \( x_n x_1 \), but \( x_n x_1 \) does not exceed \( x_n x_4 \);
- for \( n \equiv 2 \pmod{4} \) the sum \( A \) contains all the summands of \( B \) except \( x_{n-1} x_1 \) and \( x_n x_2 \), but these summands do not exceed \( x_{n-1} x_4 \) and \( x_n x_4 \);
- for \( n \equiv 3 \pmod{4} \) the sum \( A \) contains all the summands of \( B \) except \( x_{n-2} x_1, x_{n-1} x_2 \) and \( x_n x_3 \), but these summands do not exceed \( x_{n-2} x_4, x_{n-1} x_4 \), and \( x_n x_4 \).

Hence

\[ 3(x_1 + x_2 + \ldots + x_n)^2 \geq 8A \geq 8B = 8 \sum_{k=1}^{n} x_k(x_{k+1} + x_{k+2} + x_{k+3}). \]

2.3. a) Cf. [11]. By the Cauchy–Bunyakovsky inequality and Problem 2.2 we have

\[ \frac{x_1}{x_2 + x_3 + x_4} + \frac{x_2}{x_3 + x_4 + x_5} + \ldots + \frac{x_{n-1}}{x_n + x_1 + x_2} + \frac{x_n}{x_1 + x_2 + x_3} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{\sum_{k=1}^{n} x_k(x_{k+1} + x_{k+2} + x_{k+3})} \geq n \cdot \frac{3}{n}. \]

b) ???

2.4. [1, Problem 187]. We may assume that \( x_1 \leq x_2 \). Set

\[ S := x_1 + x_2 + \ldots + x_n, \quad S_1 := x_1 + x_3 + \ldots, \quad S_2 := x_2 + x_4 + \ldots. \]

Then \( S_1^2 + S_2^2 \geq (S_1 + S_2)^2 / 2 = S^2 / 2 \). Hence

\[ \frac{S^2}{2} \geq S^2 - S_1^2 - S_2^2 = 2 \sum_{(i-k)\neq 2} x_i x_k. \]

If \( n \) is even, then the last sum contains all the summands of form \( x_k x_{k+1} \). If \( n \) is odd, then the summand \( x_n x_1 \) is missing, however the sum contains a greater summand \( x_n x_2 \). So

\[ \frac{S^2}{2} \geq 2(x_1 x_2 + x_2 x_3 + \ldots + x_n x_1). \]

2.5. See the solution of 1.3 up to the inequality (4).

2.6. Induction on \( n \geq n = 4 \). Denote the left-hand side by \( L_n \). We have

\[ L_4 = \frac{x_1 + x_3}{x_2 + x_4} + \frac{x_2 + x_4}{x_1 + x_3} = a + a^{-1} \geq 2. \]

Let us prove the inductive step. We may assume that \( x_{n+1} \) is the minimal of all \( x_i \)’s. Now remove the last summand from \( L_{n+1} \), and then decrease two others. We obtain

\[ L_{n+1} \geq \frac{x_1}{x_{n+1} + x_2} + \ldots + \frac{x_n}{x_n + x_{n+1}} \geq \frac{x_1}{x_n + x_2} + \ldots + \frac{x_n}{x_{n-1} + x_n} = L_n. \]

In order to show that the constant 2 is sharp, take

\[ x_1 = x_2 = 1, \quad x_3 = t, \quad x_4 = t^2, \quad \ldots, \quad x_n = t^{n-2}. \]

When \( t \to +0 \), the first two summands tend to 1 and the remaining tends to 0.
2.7. [10]. Set $S := x_1 + x_2 + \ldots + x_n$. Use the Cauchy–Bunyakovsky inequality for sets \{$(x_k + x_{k+1})(x_k + x_{k+2})$\}. We obtain

$$\frac{x_1 + x_2}{x_1 + x_3} + \frac{x_2 + x_3}{x_2 + x_4} + \ldots + \frac{x_{n-1} + x_n}{x_{n-1} + x_1} + \frac{x_n + 1}{x_n + x_2} \geq \frac{4(x_1 + x_2 + \ldots + x_n)^2}{\sum_{k=1}^{n} (x_k + x_{k+1})(x_k + x_{k+2})}.$$  

So it suffices to prove that

$$S^2 \geq \sum_{k=1}^{n} (x_k + x_{k+1})(x_k + x_{k+2}) = \sum_{k=1}^{n} x_k^2 + 2 \sum_{k=1}^{n} x_k x_{k+1} + \sum_{k=1}^{n} x_k x_{k+2}.$$  

This can be shown by opening brackets in the left-hand side, because for $n \geq 4$ all the summands $x_k x_{k+1}$ and $x_k x_{k+2}$, where $k = 1, 2, \ldots, n$, are different.

In order to show that the constant 4 is sharp, take $x_k = a^{k-1}$ for $k = 1, 2, \ldots, n-1$ and $x_n = a^{n-2}$. When $a \to \infty$, the first $n - 3$ summands tend to 0 and the remaining summands tend to 1, 2 and 1.

Using the Cauchy–Bunyakovsky inequality as it is done in the solution of the next problem, the reader will easily find another solution of this problem reducing it to the inequality from Problem 2.4.

2.8. [6]. Use the Cauchy–Bunyakovsky inequality for sets \{$(x_k x_{k-1} + x_{k+2})$\} in \{$(x_k(x_{k-1} + x_{k+2}))$\}. We obtain

$$\frac{x_1}{x_n + x_3} + \frac{x_2}{x_1 + x_4} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_1} + \frac{x_n}{x_{n-1} + x_2} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{(x_1 x_2 + x_2 x_3 + \ldots + x_{n-1} x_1) + (x_1 x_3 + x_2 x_4 + \ldots + x_n x_2)}.$$  

So it suffices to prove that

$$S^2 \geq 3(x_1 x_2 + x_2 x_3 + \ldots + x_n x_1) + 3(x_1 x_3 + x_2 x_4 + \ldots + x_n x_2) =: 3Y,$$

where $S := x_1 + x_2 + \ldots + x_n$. Set

$$S_1 := x_1 + x_4 + \ldots, \quad S_2 := x_2 + x_5 + \ldots \quad \text{and} \quad S_3 := x_3 + x_6 + \ldots.$$  

Then $S = S_1 + S_2 + S_3$ and $S_1^2 + S_2^2 + S_3^2 \geq 2S^2$. We may assume that $x_1 \geq x_2$ and $x_2 \geq x_3$. Notice that

$$S^2 \geq \frac{3}{2}(S^2 - S_1^2 - S_2^2 - S_3^2) = 3 \sum_{i=1}^{n} x_i x_k =: 3Z.$$  

- If $n \equiv 0 \pmod{3}$, then all the summands of $Y$ are contained in $Z$.
- If $n \equiv 1 \pmod{3}$, then $Z$ contains all the summands of $Y$ except $x_n x_1$, but this summand does not exceed $x_n x_3$.
- If $n \equiv 2 \pmod{3}$, then $Z$ contains all the summands of $Y$ except $x_{n-1} x_1$ and $x_n x_2$, but these summands do not exceed $x_{n-1} x_3$ and $x_n x_3$.

Hence $S^2 \geq 3Y \geq 3Z$, which proves the initial inequality.

In order to show that the constant 3 is sharp, take $x_k = a^{k-1}$ for $k = 1, 2, \ldots, n-2$ and $x_{n-1} = x_n = 1$. When $a \to \infty$, the first and the last two summands tend to 1, while the remaining summands tend to 0.

2.9. [5]. The inequality is obtained by summing two inequalities of 2.8 (for the direct and the opposite order of variables).

In order to show that the constant 6 is sharp, take $x_k = a^{k-1}$ for $k = 1, 2, \ldots, n-2$ and $x_{n-1} = x_n = 1$. When $a \to \infty$, the last four summands tend to 1, 2, 2, 1, respectively; the remaining tend to 0.

2.10. This is conjectured in [19].

The following proof is due to P. Milošević and M. Bukić, participants of the Conference.

This inequality can be represented as sum of two inequalities for $n = 2004$ — the inequality from Problem 2.8 and the inequality

$$\frac{x_1}{x_1 + x_4} + \frac{x_2}{x_2 + x_5} + \ldots + \frac{x_n}{x_n + x_3} \geq 3.$$  

Prove the last inequality. For $n = 3m$ it is the sum of three inequalities:

$$\frac{x_1}{x_1 + x_4} + \frac{x_4}{x_4 + x_7} + \ldots + \frac{x_{n-2}}{x_{n-2} + x_1} \geq 1,$$

$$\frac{x_2}{x_2 + x_5} + \frac{x_5}{x_5 + x_8} + \ldots + \frac{x_{n-1}}{x_{n-1} + x_2} \geq 1,$$

$$\frac{x_3}{x_3 + x_6} + \frac{x_6}{x_6 + x_9} + \ldots + \frac{x_n}{x_n + x_3} \geq 1.$$
Each of these inequalities can be re-written as
\[ \frac{1}{1 + a_1} + \frac{1}{1 + a_3} + \ldots + \frac{1}{1 + a_m} \geq 1 \quad \text{where} \quad a_1 a_2 \ldots a_m = 1. \]

This can be shown by induction. The base \( m = 2 \) is the following inequality:
\[ \frac{1}{1 + a_1} + \frac{1}{1 + \frac{1}{a_1}} = 1 \geq 1. \]

To prove the induction step, let us check that
\[ \frac{1}{1 + b} + \frac{1}{1 + c} \geq \frac{1}{1 + bc}. \]

This can be done directly by reducing to a common denominator and opening brackets.

Here is the proof of A. Khrabrov. Let us prove that
\[ Z := \frac{x_1 + x_2}{x_1 + x_4} + \frac{x_2 + x_3}{x_2 + x_5} + \ldots + \frac{x_n + x_1}{x_n + x_3} \geq 6. \]

Set \( x_{3n+k} := x_k \) and, for \( r = 0, 1, 2 \),
\[ S_r := \sum_{k=1}^{n} \frac{x_{3k+r}}{x_{3k+r} + x_{3k+3+r}}, \quad X_r := \sum_{k=1}^{n} \frac{x_{3k+r}}{x_{3k+r} + x_{3k+3+r}}, \quad \text{and} \quad Y_r := \sum_{k=1}^{n} \frac{x_{3k+r+1}}{x_{3k+r} + x_{3k+3+r}}. \]

First we prove that \( X_r \geq 1 \). Consider only the case \( r = 0 \). Then
\[ X_0 S_0^2 \geq X_0 \left( \sum_{k=1}^{n} x_{3k}^2 + \sum_{k=1}^{n} x_{3k} x_{3k+3} \right) = X_0 \left( \sum_{k=1}^{n} x_{3k} (x_{3k} + x_{3k+3}) \right) \geq S_0^2, \]
where the last inequality holds by the Cauchy–Bunyakovsky inequality. So \( X_0 \geq 1 \).

Now prove that \( Y_r \geq S_{r+1}/S_r \) (we set \( S_3 := S_0 \)). Consider only the case \( r = 0 \).
\[ Y_0 S_0 S_1 \geq Y_0 \left( \sum_{k=1}^{n} x_{3k} x_{3k+1} + \sum_{k=1}^{n} x_{3k+1} x_{3k+3} \right) = Y_0 \left( \sum_{k=1}^{n} x_{3k+1} (x_{3k} + x_{3k+3}) \right) \geq S_1^2, \]
where the last inequality holds by the Cauchy–Bunyakovsky inequality. So \( Y_0 \geq S_1/S_0 \).

Summing up all the proved inequalities we obtain
\[ Z = X_0 + X_1 + X_2 + Y_0 + Y_1 + Y_2 \geq 3 + \frac{S_1}{S_0} + \frac{S_2}{S_1} + \frac{S_0}{S_2} \geq 6. \]

In order to show that the constant 6 is sharp, take \( x_1 = x_2 = x_3 = 1, x_k = a^{n-k+1} \) for \( k = 3, 4, \ldots, n \). When \( a \to 0 \), the first and the second summands tend to 2, the third and the last tend to 1, and the remaining summands tend to 0.

2.11. This proof is due to A. Khrabrov. Set \( S = x_1 + x_2 + \ldots + x_n \) and \( T = \sum_{(i-k)2} x_i x_k \). By the Cauchy–Bunyakovsky inequality for sets \( \left\{ \frac{x_k}{x_{k-1} + x_{k+3}} \right\} \) and \( \left\{ x_k (x_{k-1} + x_{k+3}) \right\} \) we have
\[ \frac{x_1}{x_{n+4}} + \frac{x_2}{x_{n+5}} + \ldots + \frac{x_{n-1}}{x_{n-2} + x_2} + \frac{x_n}{x_{n-1} + x_3} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{(x_1 x_2 + x_2 x_3 + \ldots + x_n x_1) + (x_1 x_4 + x_2 x_5 + \ldots + x_n x_3)}. \]

So it suffices to prove that
\[ S^2 \geq 4(x_1 x_2 + x_2 x_3 + \ldots + x_n x_1) + 4(x_1 x_4 + x_2 x_5 + \ldots + x_n x_3). \]

In the solution of problem 2.4 we proved that \( S^2 \geq 4T \), see (8). So it suffices to prove that
\[ T \geq (x_1 x_2 + x_2 x_3 + \ldots + x_n x_1) + (x_1 x_4 + x_2 x_5 + \ldots + x_n x_3). \]

Since \( n \) is even, all the summands of the right-hand sum are contained in the left-hand sum.

In order to show that the constant 6 is sharp, take \( x_k = a^{k-1} \) and \( k = 1, 2, \ldots, n-3 \) and \( x_{n-2} = x_{n-1} = x_n = 1 \). When \( a \to +0 \) the first summand and the three last summands tend to 1, and the remaining summands tend to 0.
2.12. [14]. Note that \( a^2 - ab + b^2 \leq \max(a, b)^2 \).

Let \( x_i \) be the maximal number of \( x_1, x_2, \ldots, x_n \). Let \( x_{i+1} \) be the maximal number of \( x_{i+1} \) and \( x_{i+2} \). Let \( x_{i+3} \) be the maximal number of \( x_{i+3} \) and \( x_{i+4} \), and so on. There exists a number \( k \) such that \( x_{ik+1} = x_{i+1} \). Hence

\[
\sum_{k=1}^{n} \frac{x_k^2}{x_{k+1}^2 - x_{k+1}x_{k+2} + x_{k+2}^2} \geq \sum_{j=1}^{k} \frac{x_{ij}^2}{x_{ij+1}^2} \geq k \geq \left[ \frac{n+1}{2} \right],
\]

where the latter inequality holds because \( k \geq n/2 \).

In order to show that the constant \( \left[ \frac{n+1}{2} \right] \) is sharp, take \( x_k = 1 \) for odd \( k \) and \( x_k = 0 \) for even \( k \). Then the left-hand side is \( \left[ \frac{n+1}{2} \right] \).

References

[20] Mordell L. J. On the inequality \( \sum_{r=1}^{n} \frac{x_r}{x_{r+1} + x_{r+2}} \geq \frac{n}{2} \) and some others // Abh. Math. Sem. Univ. Hamburg. 1958. Vol. 22. P. 229–240.