The Helly–Gallai numbers for the families of finite and quasi-finite sets

I.I. Bogdanov, G.R. Chelnokov, V.L. Dolnikov

1 Introduction

Typical problem. *In a city there are some bus routes. Every two routes have at most two common bus stops. Moreover, for each 10 routes there exist two stops A and B such that each among out ten routes passes through at least one of the stops A and B. Prove that there exist two stops such that each route in the city passes through at least one of them.*

Also show that the problem statement does not remain valid after replacing the number 10 by a smaller number. (See problem 5.4b).)

If you are familiar with this type of problems (and you like them) you may skip the preliminary text and just start working on the problems.

One of the widespread types of questions in combinatorics is the following one. Given a family $\mathcal{F}$ of sets, we need to determine whether it is possible to select few elements (usually a prescribed number of them) so that each set in $\mathcal{F}$ contains at least one selected element. (Such selection is called a *transversal* for $\mathcal{F}$.) Such questions are important, for instance, in investigating the colorings of (hyper)graphs (and thus have applications in many areas of mathematics); also such questions are very popular in convex geometry. Despite the popularity of this type of questions, many of them are still unsolved, sometimes even those looking quite innocent.

As an example, we present a known Erdös–Faber–Lovasz conjecture which is neither proved nor disproved at the present moment.

Conjecture. *In a city there are some bus routes and n bus stops. Suppose that each route passes through at least two stops, and each two routes have at most one stop in common. Then all the routes can be colored inn colors so that no two routes of the same color have a common station.*

Obviously, the necessary condition for the certain family of set to have a transversal of fixed size is that each its subfamily also has such a transversal. For some special kinds of families, it is sufficient to verify this condition for all the subfamilies of a certain size. The famous result of this type is Helly's theorem for planar convex sets (see problem 2.4); it deals with the 1-transversals.

The first our principal goal is to investigate this type of questions for the families of sets of bounded finite cardinality (section 4). The next goal is to generalize these results for the case of families of sets with some finiteness conditions for their intersections. This type of families is important, e.g., since the conditions of such kind hold for the *algebraic* sets (i.e. the sets of solutions of certain systems of algebraic equations).

The solution of some problems in this project is not known at the present moment. In this case we ask the question of the type "How far can you improve the estimate?". For such type of problems, we accept any **serial** bound (that is, a bound that works for infinite number of values of parameters). Please keep in mind that serious improvement of known results in this area is worth being published.

The problems marked by the triangle sign$^\triangledown$ *were given at the semifinal.*

2 Introduction to the Helly numbers

2.1. Consider a family of segments in a line. Suppose that the intersection of all segments of this family is empty. Prove that there exist two segments in this family, such that their intersection is empty.

2.2. Consider a tree G (i.e. a connected graph without cycles). Consider the finite family of connected subgraphs of G such that the intersection of all these subgraphs is empty. Prove that there exist two subgraphs in this family, such that their intersection is empty.

2.3. Consider a finite family of arcs of some fixed circle. Suppose that every 1000 arcs of this family have a nonempty intersection. Show that all the arcs of the family do not necessarily have a nonempty intersection.

All the three problems above deal with the same property of some family of sets; for instance, in problem 2.1 we work with the family of all segments in the line. Now we define the property.

Definition 1. *Let $\mathcal{F}$ be some (not necessarily finite) family of sets. A positive integer k is called a* Helly number *for the family $\mathcal{F}$ if the following statement holds:*

Let $\mathcal{G}$ be an arbitrary finite subfamily of $\mathcal{F}$. Suppose that the intersection of all sets in $\mathcal{G}$ is empty. Then there exist at most k sets in $\mathcal{G}$ such that their intersection is empty.

In other words, k is a Helly number of the set $\mathcal{F}$ if the following “Helly type statement” takes place:

Let $\mathcal{G}$ be an arbitrary finite subfamily of $\mathcal{F}$. Assume that for every k sets in $\mathcal{G}$, their intersection is nonempty. Then the intersection of all the sets in $\mathcal{G}$ is also nonempty.

Using the introduced notation, one can reformulate the problems 1.1–1.2 as follows: number 2 is a Helly number for both the family of the segments in a line and the family of all subtrees of some fixed tree. The statement of problem 1.3, after a similar reformulation, claims that there is no Helly number for the family of all he arcs of a fixed circle.

Definition 2. *Let $\mathcal{F}$ be a family of sets. We write $H(\mathcal{F}) = k$ if k is the minimal Helly number for the family $\mathcal{F}$ (provided that there exists some Helly number for $\mathcal{F}$). If there is no Helly number for $\mathcal{F}$, then we write $H(\mathcal{F}) = \infty$.*

The next problems will help you to get acquainted with the Helly numbers.

2.4. (Helly's theorem for the convex sets in the plane.) Let $\mathcal{C}$ be the family of all convex sets in the plane. Prove that $H(\mathcal{C}) = 3$.

Remark. For the problem above, the condition that the family $\mathcal{G} \subset \mathcal{C}$ is finite (in Definition 1) may be replaced with the condition that all sets of $\mathcal{C}$ are closed and bounded.

2.5. Let $\mathcal{P}$ be the family of all infinite increasing arithmetic progressions consisting of positive integers. Prove that $H(\mathcal{P}) = 2$.

2.6. Let $\mathcal{O}$ be the set of all the circumferences in the plane. Find $H(\mathcal{O})$. (**Reminder.** A circumference is a boundary of a disk in the plane.)

3 Introduction to the Helly–Gallai numbers

We need a bit more general concept than the Helly numbers.

Definition 3. *Let $\mathcal{F}$ be a family of sets. A set X is* a transversal *for the family $\mathcal{F}$ if $X \cap A \neq \varnothing$ for all $A \in \mathcal{F}$. A transversal X is called a t-*transversal *if $|X| = t$.*

Remark. For any family $\mathcal{F}$, the existence of a 1-transversal is equivalent to the condition that the intersection of all sets of $\mathcal{F}$ is nonempty.

Definition 4. *Let $\mathcal{F}$ be a family of sets. A positive integer k is called a t-*Helly–Gallai number *for $\mathcal{F}$ if the following statement holds:*

Let $\mathcal{G}$ be an arbitrary finite subfamily of $\mathcal{F}$. Suppose that there is no t-transversal for $\mathcal{G}$. Then there exists a subfamily of at most k sets in $\mathcal{G}$ also admitting no t-transversal.

Next, if there exists a t-Helly–Gallai number for a family $\mathcal{F}$, then we will denote by $HG_t(\mathcal{F})$ the minimal such number. Otherwise we write $HG_t(\mathcal{F}) = \infty$.

Remark. Note that by definition we have $H(\mathcal{F}) = HG_1(\mathcal{F})$, so the new concept is a generalization of the Helly numbers introduced above..

Again, the next problems aim to help to get acquainted with a new notion.

3.1. Let $\mathcal{S}$ be the family of segments in the line.
a) Prove that $HG_2(\mathcal{S}) = 3$
b) Prove that $HG_t(\mathcal{S}) = t + 1$.

3.2. a) Construct a family $\mathcal{F}$ such that $H(\mathcal{F}) = 2$ but $HG_2(\mathcal{F}) \geq 1000$.
b) Prove that there exists a family $\mathcal{F}$ such that $H(\mathcal{F}) = 2$ but $HG_2(\mathcal{F}) = \infty$.

3.3. Prove that $HG_2(\mathcal{C}) = \infty$. (Recall that $\mathcal{C}$ is the family of all convex sets in the plane.)

3.4*. Denote by $\mathcal{L}$ the family of all the lines in the plane.
a) Prove that $HG_t(\mathcal{L}) \leq t^2 + 1$ for all $t \geq 3$.
b) How far can you improve this bound?

4 Families of sets of bounded cardinality and their Helly–Gallai numbers

One can see from the previous section that the existence of a finite Helly number for some family $\mathcal{F}$ does not necessarily imply that the Helly–Gallai numbers of this family are finite. Thus, one needs some additional conditions on the family $\mathcal{F}$. One of such natural conditions is the boundedness of all the cardinalities of the sets in $\mathcal{F}$.

Definition 5. *Denote by $\mathcal{N}_d$ the family of all sets having at most d elements each.*

4.1. Let $\mathcal{G}$ be some finite family of sets having at most d elements each. Suppose that every $d+1$ sets of $\mathcal{G}$ have a nonempty intersection. Prove that the intersection of all the sets of $\mathcal{G}$ is nonempty.
Moreover, show that in the previous statement, the number $d + 1$ can not be replaced with d.
Comprehension exercise. Reformulate this problem with the use of terminology introduced above.

4.2. Prove that all the numbers of the form $HG_t(\mathcal{N}_d)$ are finite.

4.3. a) Prove that $HG_t(\mathcal{N}_2) \geq C^2_{t+2}$.
b) Prove that $HG_t(\mathcal{N}_2) = C^2_{t+2}$.

4.4. Prove that $HG_t(\mathcal{N}_d) \geq C^t_{d+t}$.

4.5*. (The principal result of this section) Prove that $HG_t(\mathcal{N}_d) = C^t_{d+t}$.

The problem above is tough. The next problem may be used as a hint.

4.6. a) (Katona's problem) Assume that the sets $A_1, A_2, \dots, A_n, B_1, \dots, B_n$ are chosen so that $|A_i| = d$, $|B_i| = t$, for $i \neq j$ the set A_i has at least one common element with the set B_j, but $A_i \cap B_i = \varnothing$. Prove that $n \leq C^t_{d+t}$.
b) Assuming that Katona's problem is solved, prove the problem 4.5.

4.7.$^\triangledown$ Assume that the sets $A_1, A_2, \dots, A_n, B_1, \dots, B_n$ satisfy *Katona's conditions*, i.e. $A_i \cap B_i = \varnothing$ and $A_i \cap B_j \neq \varnothing$ for all $i \neq j$. Let $a_i = |A_i|$, $b_i = |B_i|$. Prove that $\displaystyle\sum_{i=1}^{n} \frac{1}{C^{a_i}_{a_i+b_i}} \leq 1$.

The previous problem presents the constraints for the cardinalities of the sets satisfying Katona's conditions. The next problem shows that this constraint is far from being sufficient.

4.8.$^\triangledown$ Prove that there exist positive integers $a_1, \dots, a_n, b_1, \dots, b_n$ such that (i) $\displaystyle\sum_{i=1}^{n} \frac{1}{C^{a_i}_{a_i+b_i}} < \frac{1}{10^{100}}$, but (ii) there are no sets $A_1, A_2, \dots, A_n, B_1, \dots, B_n$ satisfying Katona's conditions such that $|A_i| = a_i$, $|B_i| = b_i$.

5 The Helly–Gallai numbers for the quasi-finite sets

In this section we weaken the conditions for the families considered.

Definition 6. *Let $\mathcal{F}$ be a family of sets. We say that $\mathcal{F}$ is a family of degree d if for every two distinct sets $A, B \in F$ we have $|A \cap B| \leq d$.*

Now let us fix arbitrary positive integers d and t. Next, for every family $\mathcal{F}$ of degree d let us find its (minimal) t-Helly–Gallai number $HG_t(\mathcal{F})$. Denote by $a(d;t)$ the maximal among these numbers.

In other words, $a(d;t)$ is a minimal number which is a Helly–Gallai number for every family $\mathcal{F}$ of degree d.

This section is devoted to different bounds for the numbers $a(d;t)$;

5.1. Let $\mathcal{F}$ be a family of sets. Suppose that $|A \cap B| = 1$ for every distinct $A, B \in \mathcal{F}$ (in particular, this condition implies that $\mathcal{F}$ is a family of degree 1). Suppose, in addition, that among any four sets in $\mathcal{F}$, there exists three sets having a nonempty intersection. Prove that one can delete one of the sets from $\mathcal{F}$ so that all the remaining sets will have a nonempty intersection.

5.2. Let $\mathcal{F}$ be a family of sets. Suppose that $\mathcal{F}$ contains at least 17 sets, and for every distinct $A, B \in \mathcal{F}$ we have $|A \cap B| = 1$. Moreover, suppose that among any 5 sets in $\mathcal{F}$, there are three sets with a nonempty intersection. Prove that there exists a 2-transversal for $\mathcal{F}$.

5.3. Prove that $a(d;1) = d+2$.

5.4. a) Prove that $a(1;2) = 6$.
b) Prove that $a(2;2) = 10$.
c) Prove that $a(3;2) = 15$.
d) Prove that $a(4;2) = 21$.
e)* Try to prove the analogous statement for some larger values of n.

5.5. a) Prove that $a(d;2) \geq C_{d+3}^2$.
b) Prove that $a(d;2) \leq 2d^2 + 3$ for all $d \geq 2$.
c)* How far can you improve these bounds?

Remark. In the problem above, the authors can improve the upper bound (that from part b)), but still there is a substantial gap between the upper and the lower bounds. It would be very interesting to obtain upper and lower bounds which are asymptotically the same; that is, their ratio should tend to 1 as $d \to \infty$.

5.6. a) Prove that $a(1;3) = 10$.
b) Prove that $a(1;4) = 15$.
c)* For which values of t you can prove the analogous statement?

5.7. a) Prove that $a(1;t) \geq C_{t+2}^2$.
b) Prove that $a(1;t) \leq t^2 + 1$ for all $t \geq 3$.
c)* How far can you improve these bounds?

5.8. a) Prove that $a(d;t) \geq C_{d+t+1}^t$.
b)** Do there exist the values of d and t such that the inequality in a) is strict?

Remark. The authors do not know the answer for 5.8б).

5.9. a) Prove that the number $a(d;t)$ is finite for every pair (d,t).
b) Try to obtain a good[1] bound for the number $a(d;t)$.
The remaining part of the text was given at the semifinal.

The next notion for the quasi-finite sets is analogous to Katona's conditions.

Definition 7. *Let $\mathcal{G}$ be a finite family of sets. We say that $\mathcal{G}$ is* $(d;t)$-exceptional *if (i) $\mathcal{G}$ is a family of degree d, and (ii) for every $A \in \mathcal{G}$ there exists a set X_A with $|X_A| = t$ such that $X_A \cap A = \varnothing$ but $X_A \cap B \neq \varnothing$ for all $B \in \mathcal{G}$ different from A.*

Denote by $b(d;t)$ the maximal cardinality of a $(d;t)$-exceptional family. (We set $b(d;t) = \infty$ if there exist $(d;t)$-exceptional families of an arbitrary large cardinality.)

[1] as good as possible...

5.10.$^\triangledown$ Prove that $b(d;t) \geq a(d;t)$.

5.11.$^\triangledown$ Prove that the number $b(d;t)$ is finite for all d and t.

The next problems in this section are devoted to the investigation of exceptional families.

From now on, we fix positive integers d and t; we also fix some $(d;t)$-exceptional family $\mathcal{G}$. Recall that for every $A \in \mathcal{G}$ we fix one set X_A such that $|X_A| = t$, $X_A \cap A = \varnothing$, but $X_A \cap B \neq \varnothing$ for all $B \in \mathcal{G}$ distinct from A.

Definition 8. *Let x be some element. Denote by $g(x)$ the number of the sets $A \in \mathcal{G}$ such that $x \in A$. Denote by $h(x)$ the number of the sets $A \in \mathcal{G}$ such that $x \in X_A$.*

5.12.$^\triangledown$ a) Assume that $t = 2$. Prove that $h(x) \leq d + 2$ for every x.

b) Prove that $h(x) \leq b(d;t-1)$ (for an arbitrary t).

5.13.$^\triangledown$ Prove that $g(x) \leq b(d-1;t)$ for every x.

5.14.$^\triangledown$ Assume that $t = 2$ and $h(x) \leq d$. Prove that $|\mathcal{G}| \leq g(x) + h(x) + b(d - h(x); 2)$.

6 The Helly–Gallai numbers for the two-dimensional quasi-finite sets

The families considered in the previous section may be regarded as *1-dimensional* since the intersection of any two of them is finite. In particular, setting $t = 1$ we get a combinatorial analogue of a family of lines. In the same way, we may define *2-dimensional* families as follows.

Definition 9. *Let $\mathcal{F}$ be a family of sets. Let $\mathcal{F}'$ be the family of all the intersections of two distinct elements from $\mathcal{F}$, i.e.*

$$\mathcal{F}' = \{A \cap B \ : \ A, B \in \mathcal{F}, \ A \neq B\}.$$

We say that $\mathcal{F}$ is a 2-dimensional family of degree d *if $\mathcal{F}'$ is a (1-dimensional) family of degree d.*

Now let us fix some positive integers d and t. For every 2-dimensional family $\mathcal{F}$ of degree d, consider its (minimal) t-Helly–Gallai number $HG_t(\mathcal{F})$. Denote by $a_2(d;t)$ the maximal such number.

In this section, we do not propose an explicit investigation agenda. Instead of that, we propose you to investigate the numbers $a_2(d;t)$ by yourself. We will accept all the results about these numbers. The problems below mark only some natural directions of investigation.

6.1.$^\triangledown$ Prove that all the numbers $a_2(d;t)$ are finite.

6.2.$^\triangledown$ Assume that $d = 0$. Give the necessary definitions on your own and prove that $g(x) \leq t + 1$.

6.3.$^\triangledown$ (INVESTIGATORY, "WITH NO DOUBLE INTERSECTIONS") Find some upper and lower bounds for the numbers $a(0;t)$ for some values of t.

6.4.$^\triangledown$ (INVESTIGATORY, "THE CASE OF PLANES") Find some upper and lower bounds for the numbers $a(1;t)$ for some values of t.

6.5.$^\triangledown$ (INVESTIGATORY, A GENERAL CASE) Find some upper and lower bounds for the numbers $a(d;t)$ for some values of d and t.

Remark. In the last three problems, as usual, serial bounds are more valuable than partial results for fixed values of $(d;t)$. Nevertheless, we insistently recommend you to start with a consideration of a sufficient amount of particular cases.