The Helly–Gallai numbers for the families of finite and quasi-finite sets.

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Solutions.

2.1. *Consider a family of segments in a line. Suppose that the intersection of all segments of this family is empty. Prove that there exist two segments in this family, such that their intersection is empty.*

Denote the given segments as $[a_1, b_1], \dots, [a_n, b_n]$. Let $a_s = \max_{1 \le i \le n} a_i$, $b_t = \min_{1 \le i \le n} b_i$ (it is possible that $s = t$). If $a_s \le b_t$ then each given segment contains a_s, which contradicts the condition of the problem. Hence $a_s > b_t$, therefore $t \ne s$, and the sth segment has no common point with the tth one.

2.2. *Consider a tree G (i.e. a connected graph without cycles). Consider the finite family of connected subgraphs of G such that the intersection of all these subgraphs is empty. Prove that there exist two subgraphs in this family, such that their intersection is empty.*

If we delete any edge e from a tree G, we cut this tree into two components. Moreover, if a subtree G' does not contain e, then it lies in one of these components.

Denote our subtrees by $G_1, \dots, G_n$. Let k be a maximal integer such that the intersection $G = G_1 \cap G_2 \cap \dots \cap G_k$ is nonempty (note that $k < n$). Then the intersection of G and G_{k+1} is empty.

Now consider a shortest path conecting G with G_{k+1}. The first edge e of this path does not belong to G, hence some G_i (with $1 \le i \le k$) also does not contain e. Let us delete this edge. The subtrees G_{k+1} and G come to different components of the remaining graph. Therefore G_{k+1} and G_i are also in different components. Hence their intersection is empty.

2.3. *Consider a finite family of arcs of some fixed circle. Suppose that every 1000 arcs of this family have a nonempty intersection. Show that all the arcs of the family do not necessarily have a nonempty intersection.*

Partition the circle into 1001 arcs (we call them *small* arcs). We call a complement of any small arc a *large* arc. We claim that the family of large arcs is a desired example. Namely, it is easy to check that the intersection of any 1000 large arcs is nonempty, but the intersection of all of them is obviously empty.

Remark. The large arcs in the solution are assumed not to contain their endpoints. One may decrease them slightly to obtain an example consisting of closed arcs.

2.4. *(*HELLY'S THEOREM FOR THE CONVEX SETS IN THE PLANE.*) Let $\mathcal{C}$ be the family of all convex sets in the plane. Prove that $H(\mathcal{C}) = 3$.*

Induction on the number $n \ge 4$ of the considered convex sets. The base case is $n = 4$. Consider our convex sets F_1, F_2, F_3, F_4. Let x_i be a point of intersection of all of them except F_i. We apply the following lemma.

Lemma 1 (Radon's theorem)**.** *One can partition any four points in the plane into two sets so that the convex hulls of these sets have a common point.*

Proof. Consider two cases: (i) the points are the vertices of some convex quadrilateral, and (ii) one of the points lies inside the triangle formed by the other three. □

So, suppose for example that the convex hulls of the sets $\{x_1, x_2\}$ and $\{x_3, x_4\}$ intersect at point x. Then x lies if F_3 and F_4, since the whole segment x_1x_2 lies in these sets. Analogously, $x \in F_1 \cap F_2$. All the other cases are completely analogous.

For the induction step, assume that $n \geq 4$ and consider $n+1$ convex sets $F_1, \ldots F_{n+1}$. Let us construct a new family consisting of n convex sets $F'_1 = F_1 \cap F_{n+1}, \ldots, F'_n = F_n \cap F_{n+1}$. By the statement of the base case, each three of the new sets have a nonempty intersection. Then, by the induction hypothesis, all the new sets have a common point, QED.

2.5. *Let $\mathcal{P}$ be the family of all infinite increasing arithmetic progressions consisting of positive integers. Prove that $H(\mathcal{P}) = 2$.*

Let $\mathcal{G}$ be a finite family of progressions such that every two of them have a common element. Note that the intersection of each two progressions from $\mathcal{G}$ is infinite (in fact, it is an infinite progression as well). So, with no loss of generality we may assume that the ith progression is defined by the condition $x \equiv a_i \pmod{n_i}$ (for that, we need only to "forget" about some initial segment of the row of the positive integers).

Consider one of our progressions; say it is defined by $x \equiv a \pmod{n}$. This condition is equivalent to the system of the relations having the form $x \equiv a \pmod{p_j^{\alpha_j}}$ (here, $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ is the prime decomposition of n). Now, let us write down such a system for each progression. We need to prove that the system of all obtained relations is collocated (i.e. has a solution).

Note that every two relations in our system are collocated, since they correspond either to one or to two progressions. Next, suppose that the system contains two equivalences modulo the powers of the same prime p (say $x \equiv a \pmod{p^\alpha}$ and $x \equiv b \pmod{p^\beta}$). These relations are collocated; we may assume that $\alpha \geq \beta$. Then the latter relation is a consequence of the former one, so we may delete the latter from the system.

Performing such steps, we finally arrive to a system where all moduli are the powers of distinct primes. Such a system has a solution by the Chinese Remainder Theorem.

Remark. One may show that for every $t \geq 2$, there are no t-Helly–Gallai numbers for the set of out progressions,

2.6. *Let $\mathcal{O}$ be the set of all the circumferences in the plane. Find $H(\mathcal{O})$.*

Answer. $H(\mathcal{O}) = 4$.

Firstly, we prove that 4 is a Helly number for $\mathcal{O}$. Consider a finite family $\mathcal{G}$ of circumferences such that the intersection of all of them is empty. Consider arbitrary different circumferences $O_1, O_2 \in \mathcal{G}$. They have at most two points of intersection. Denote these points by A and B (if some of these points do not exist, the arguments are the same).

Recall that no point belongs to all the circumferences. Therefore there exist $O_3, O_4 \in \mathcal{G}$ such that $A \notin O_3$ and $B \notin O_4$. Hence O_1, O_2, O_3, O_4 have no common points, as desired.

We are left to present an example showing that $H(\mathcal{O}) > 3$. Consider a noncyclic nondegenerate quadrilateral $ABCD$. Consider four circumcircles of the triangles ABC, ABD, ACD, BCD. Each three of them have a nonempty intersection but all four do not.

3.1. *Let $\mathcal{S}$ be the family of segments in the line.*

a) Prove that $HG_2(\mathcal{S}) = 3$

b) Prove that $HG_t(\mathcal{S}) = t + 1$.

Item a) is a particular case of b); so we present only a solution for the latter one.

Denote the segments under consideration as $[a_1, b_1], \ldots, [a_n, b_n]$. Denote by c_1 the minimal number among all b_i's. Then each our segment either contains c_1 or lies to the right of c_1. Denote by c_2 the minimal b_i such that the segment $[a_i, b_i]$ lies to the right of c_1. Analogously, we get that each our segment either contains c_1 or c_2, or lies to the right of c_2.

Proceeding in the same way while it is possible, we finally get the sequence $c_1, c_2, \ldots, c_k$ such that each segment contains at least one of these k numbers. So, if $k \leq t$ then we have found a desired transversal. Assume now that $k \geq t + 1$. Consider $t + 1$ segments with the right ends $c_1, \ldots, c_{t+1}$; by our choice, they are pairwise disjoint, so they do not admit a t-transversal.

Thus, we have proved that either all the segments have a t-transversal, or there are $t+1$ pairwise disjoint segments. This means that $t + 1$ is a t-Helly–Gallai number for the segments in the line. Finally, an obvious example of $t + 1$ pairwise disjoint segments shows that $HG_t(\mathcal{S}) > t$.

3.2. *a) Construct a family $\mathcal{F}$ such that $H(\mathcal{F}) = 2$ but $HG_2(\mathcal{F}) \geq 1000$.*

b) Prove that there exists a family $\mathcal{F}$ such that $H(\mathcal{F}) = 2$ but $HG_2(\mathcal{F}) = \infty$.

a) Set $k = 500$. We construct a family $\mathcal{F}$ of $2k+1$ sets $A_1, \dots, a_{2k+1}$ (all indices are considered modulo $2k+1$; thus $A_i = A_{2k+1+i}$) as follows. Firstly, consider a set of indices $S \subset \{1, 2, \dots, 2k+1\}$ which does not contain two indices with difference 1 (in particular, we prohibit the situation $1, 2k+1 \in S$). For each such set S, we introduce an element x_S (the elements for different sets should be different as well). Next, we define $A_k = \{x_S : k \in S\}$.

The obtained family has a Helly number 2. Indeed, assume that a subfaily $\mathcal{G} \subset \mathcal{F}$ has an empty intersection; let $\mathcal{G} = \{A_i : i \in I\}$. If I contains two indices i and $i+1$ with difference 1, then $A_i, A_{i+1} \in \mathcal{G}$ and $A_i \cap A_{i+1} = \varnothing$. Otherwise all the sets in $\mathcal{G}$ contain x_I which is impossible.

Finally, we claim that every $2k$ sets from $\mathcal{F}$ have a 2-transversal while all $2k+1$ sets do not. Assume that $\mathcal{F}$ admits a 2-transversal $\{x_S, x_T\}$ with $|S| \geq |T|$. Then $|S| \geq k+1$, and hence S contains two neighboring indices which is impossible. On the other hand, let us delete A_{2k+1} from $\mathcal{F}$; all the other sets have a 2-transversal $\{x_S, x_T\}$ where $S = \{1, 3, \dots, 2k-1\}$ and $T = \{2, 4, \dots, 2k\}$.

Thus, $HG_2(\mathcal{F}) = 2k+1$.

b) For every positive integer k, let us construct a family $\mathcal{F}_k$ as above. We may assume that the sets from different families have no common elements. Then it is easy to show that the union $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots$ satisfies all the requirements.

3.3. *Prove that $HG_2(\mathcal{C}) = \infty$. (Recall that $\mathcal{C}$ is the family of all convex sets in the plane.)*

It suffices to prove that $HG_2(\mathcal{C}) \geq 2k+1$ for every positive integer k. Consider a circle with center O; let $x_1, \dots, x_{2k+1}$ be the vertices of a regular $(2k+1)$-gon inscribed into this circle (again, we regard the indices modulo $2k+1$). Let A_i be a convex hull of the points $x_i, \dots, x_{i+k-1}$. Then all the considered sets except any of them admit a 2-transversal (e.g., $\{x_k, x_{2k}\}$ is a 2-transversal for all the sets except A_{2k+1}).

On the other hand, we claim that each point T belongs to not more than k sets. Actually, let the ray OT intersect the boundary of $(2k+1)$-gon on the semiinterval $(x_{i-1}, x_i]$. Then T may belong only to the sets $A_{i-k+1}, \dots, A_i$ as desired.

Hence our family of cardinality $(2k+1)$ does not admin a 2-transversal, QED.

3.4*. *Denote by $\mathcal{L}$ the family of all the lines in the plane.*

a) Prove that $HG_t(\mathcal{L}) \leq t^2 + 1$ for all $t \geq 3$.

b) How far can you improve this bound?

a) Induction on $t \geq 3$. We first prove the induction step; the base case $t = 3$ is investigated further.

Let $\mathcal{G}$ be a finite family of lines such that every $\leq t^2+1$ lines from $\mathcal{G}$ have a t-transversal. Choose arbitrary t^2+1 lines from $\mathcal{G}$. One of the points of their t-transversal should belong to at least $t+1$ chosen lines. Denote this point by A, and let $\ell_1, \dots, \ell_{t+1} \in \mathcal{G}$ be $t+1$ lines containing A.

Denote by $\mathcal{G}'$ a subfamily of $\mathcal{G}$ consisting of those lines which do not contain A. We claim that each subfamily $D \subseteq \mathcal{G}$ such that $|D| \leq t^2 - t$ has a $(t-1)$-transversal. Actually, we know that the family $D \cup \{\ell_1, \dots, \ell_{t+1}\}$ has a t-transversal X. By the pigeonhole principle, two of the lines $\ell_1, \dots, \ell_{t+1}$ should pass through the same point of X; hence this point is A, i.e. $A \in X$. Finally, since the lines from D do not contain A, the set $X \setminus \{A\}$ is a transversal of D.

Note now that $t^2 - t \geq (t-1)^2 + 1 \geq HG_{t-1}(\mathcal{L})$. Hence $\mathcal{G}'$ has a $(t-1)$-transversal X by the induction hypothesis; therefore $X \cup \{A\}$ is a desired t-transversal for $\mathcal{G}$.

The proof of the base case follows the same lines except for the last paragraph (namely, the only argument where we use the induction hypothesis). To finish this argument, it suffices to show that $t^2 - t = 6 \geq HG_2(\mathcal{L})$.

Note first that $HG_1(\mathcal{L}) \leq 3$. Actually, if every three lines from $\mathcal{G}$ have a common point, then the intersection point of two of these lines should belong to all the other lines. Finally, the inequality $HG_2(\mathcal{L}) \leq 6$ can be obtained again in the same way as the induction step (with the numbers t^2+1 and $t^2 - t$ replaces by 6 and 3, respectively).

b) We prove that $HG_t(\mathcal{L}) = C_{t+2}^2$.

Firstly, we show that $HG_t(\mathcal{L}) \geq C^2_{t+2}$. Take $t+2$ points $T_1, \dots, T_{t+2}$ in a general position, and draw all the lines connecting them. One may show that it is possible to choose the points in such a way that no three lines intersect at a point different from T_i's. For such arrangement of points, our C^2_{t+2} lines do not admit a t-transversal, but all the lines except an arbitrary one have a t-transversal.

It remains to prove that $HG_t(\mathcal{L}) \leq C^2_{t+2}$. We will use the following lemma.

Lemma 2. *Let $P_1(x,y), \dots, P_k(x,y)$ be the polynomials in two variables of degree $\leq t$. Suppose that $k > C^2_{t+2}$. Then there exist the numbers $s_1, \dots, s_k$ (not all of them are zeroes) such that $s_1P_1(x,y) + \dots + s_kP_k(x,y) = 0$.*

Proof. Our conditions on the variables s_i can be rewritten as a system of C^2_{t+2} linear homogeneous (i.e. with zero constant terms) equations (the number of equations is simply the number of monomials of degree $\leq t$ in two variables). Since the number of variables is strictly greater than the number of equations, this system has a nonzero solution. □

Now we return to the solution. Obviously, it suffices to prove the following statement: If $n \geq C^2_{t+2}$, and every n lines in $\mathcal{G}$ have a t-transversal, then every $n+1$ lines from $\mathcal{G}$ also have a t-transversal.

Assume the contrary and consider a family $\mathcal{F}$ of $n+1$ lines violating our statement. Introduce a Cartesian system so that the origin does not lie on any considered line. For an arbitrary index j, the equation of the jth line ℓ_j can be written in the form $a_jx + b_jy + 1 = 0$.

Now let us fix an arbitrary index i for a while. All the lines in $\mathcal{F}$ except ℓ_i have a t-transversal consisting of points (x_1, y_1), $\dots$, (x_t, y_t) (surely, none of these points lie on ℓ_i). This means that $\prod_{k=1}^t (a_jx_k + b_jy_k + 1) = 0$ for all $j \neq i$, but the same relation is false for $j = i$. Denoting now $P_i(a,b) = \prod_{k=1}^t (ax_k + by_k + 1)$, we obtain that $P_i(a_j, b_j) = 0$ for all $j \neq i$, but $P_i(a_i, b_i) \neq 0$.

All the obtained polynomials have degree t. By the lemma above, there exist the numbers $s_1, \dots, s_n$ such that $\sum_{i=1}^n s_iP_i(a,b) = 0$. We may assume that $s_1 \neq 0$. Hence we get $\sum_{i=1}^n s_iP_i(a_1, b_1) = s_1P_1(a_1, b_1) \neq 0$. A contradiction.

4.1. *Let $\mathcal{G}$ be some finite family of sets having at most d elements each. Suppose that every $d+1$ sets of $\mathcal{G}$ have a nonempty intersection. Prove that the intersection of all the sets of $\mathcal{G}$ is nonempty.*

Moreover, show that in the previous statement, the number $d+1$ can not be replaced with d.

Comprehension exercise. *Reformulate this problem with the use of terminology introduced above.*

The problem statement can be rewritten as $H(\mathcal{N}_d) = d+1$.

Let $\mathcal{G}$ be a family of all subsets with d elements in some fixed set X of cardinality $d+1$. Then each d elements have a common element but the whole family does not. Thus, $H(\mathcal{N}_d) \geq d+1$.

Consider now an arbitrary finite subfamily $\mathcal{G} \subset \mathcal{N}_d$ such that each $d+1$ sets in $\mathcal{G}$ share a common element. For every $i = 1, \dots, d+1$, denote by s_i the minimal cardinality of the intersection of some i sets from $\mathcal{G}$. Obviously, $1 \leq s_{d+1} \leq s_d \leq \dots \leq s_1 = d$. It follows that $s_{j+1} = s_j$ for some $j \in [1, d]$.

Consider now the sets $A_1, \dots, A_j \in \mathcal{G}$ such that $|Q| = s_j$, where $Q = A_1 \cap \dots \cap A_j$. Assume that $Q \not\subseteq A$ for some $A \in \mathcal{G}$; then we get $s_{j+1} = s_j > |Q \cap A| = |A_1 \cap \dots \cap A_j \cap A|$ which contradicts to the definition of s_{j+1}. Thus, all the sets in $\mathcal{G}$ contain a nonempty set Q, QED.

4.2. *Prove that all the numbers of the form $HG_t(\mathcal{N}_d)$ are finite.*

Surely, this problem is a corollary of the problem 4.5. Here we present an easier proof.

Induction on t. The base case $t = 1$ follows from the previous problem.

For the induction step, assume that $HG_{t-1}(\mathcal{N}_d) = S < \infty$. We prove that $HG_t(\mathcal{N}_d) \leq T = S + C^t_{Sd}$.

Consider an arbitrary finite subfamily $\mathcal{G} \subset \mathcal{N}_d$ such that every T sets in $\mathcal{G}$ have a t-transversal. If every S sets of $\mathcal{G}$ have $(t-1)$-transversal, then the family $\mathcal{G}$ has even a $(t-1)$-transversal by the induction hypothesis. Otherwise, there exists a subfamily $\mathcal{K} = \{A_1, \dots, A_S\} \subset \mathcal{G}$ which has no $(t-1)$-transversal. Thus each t-transversal of the family $\mathcal{K}$ should be contained in the set $Q = A_1 \cup \dots \cup A_S$. Remark that $|Q| \leq Sd$.

Assume now that the family $\mathcal{G}$ has no t-transversal. Then for each t-element subset $X \subset Q$ there exists a set $A_X \in \mathcal{G}$ such that $A_X \cap X = \varnothing$. Now consider the subfamily

$$\mathcal{K}' = \mathcal{K} \cup \{A_X \ : \ X \subset Q, \ |X| = t\}.$$

We have $|\mathcal{K}'| \le S + C^t_{Sd} = T$. Therefore $\mathcal{K}'$ must have a t-transversal Y. Since $\mathcal{K} \subset \mathcal{K}'$, we have $Y \subset Q$. Finally, by the choice of $\mathcal{K}'$, there exists the set $A_Y \in \mathcal{K}'$ such that $A_Y \cap Y = \varnothing$. This contradiction concludes the proof.

4.3. *a) Prove that $HG_t(\mathcal{N}_2) \ge C^2_{t+2}$.*
b) Prove that $HG_t(\mathcal{N}_2) = C^2_{t+2}$.

a) See problem 4.4.
b) Follows from problem 4.5.

4.4. *Prove that $HG_t(\mathcal{N}_d) \ge C^t_{d+t}$.*

It suffices to take a family $\mathcal{G}$ of all d-element subsets of some fixed $(d+t)$-element set X. Actually, each t-element subset in X has a common element with each set in $\mathcal{G}$ except one.

4.5*. *(*THE PRINCIPAL RESULT OF THIS SECTION*) Prove that $HG_t(\mathcal{N}_d) = C^t_{d+t}$.*

See the next problem.

4.6. *a) (Katona's problem) Assume that the sets $A_1, A_2, \dots, A_n, B_1, \dots, B_n$ are chosen so that $|A_i| = d$, $|B_i| = t$, for $i \ne j$ the set A_i has at least one common element with the set B_j, but $A_i \cap B_i = \varnothing$. Prove that $n \le C^t_{d+t}$.*
b) Assuming that Katona's problem is solved, prove the problem 4.5.

a) Follows immediately from problem 4.7.
b) Assume the contrary. Consider a family $\mathcal{G} \subset \mathcal{N}_d$ of minimal cardinality such that every its subfamily with $\le C^t_{d+t}$ sets has a t-transversal, but $\mathcal{G}$ does not have a t-transversal. Surely, we have $|\mathcal{G}| > C^t_{d+t}$.

Let $\mathcal{G} = \{A_1, \dots, A_n\}$. By our choice, for every $i = 1, \dots, n$ the family $\mathcal{G} \setminus \{A_i\}$ has a t-transversal B_i. By the choice again, we have $A_i \cap B_i = \varnothing$. Thus, the sets $A_1, \dots, A_n, B_1, \dots, B_n$ satisfy the conditions of Katona's problem and hence $|\mathcal{G}| = n \le C^t_{d+t}$. A contradiction.

4.7. *Assume that the sets $A_1, A_2, \dots, A_n, B_1, \dots, B_n$ satisfy* Katona's conditions, *i.e. $A_i \cap B_i = \varnothing$ and $A_i \cap B_j \ne \varnothing$ for all $i \ne j$. Let $a_i = |A_i|$, $b_i = |B_i|$. Prove that $\sum_{i=1}^{n} \frac{1}{C^{a_i}_{a_i+b_i}} \le 1$.*

Let $X = A_1 \cup \dots \cup A_n \cup B_1 \cup \dots \cup B_n$; denote $n = |X|$. Consider all $m!$ ways of numbering the elements of X by $1, \dots, m$ (or, equivalently, all bijections $\sigma : X \to \{1, 2, \dots, m\}$). Consider any such numbering σ; we say that it is *of ith type* (for some $1 \le i \le n$) if all the elements of A_i have smaller numbers than all the elements of B_i (i.e. $\sigma(a) < \sigma(b)$ for all $a \in A_i$, $b \in B_i$).

Firstly, we note that each numbering is of at most one type. Assume, to the contrary, that σ is of both ith and jth types for $i \ne j$. Denote $a_i = \max_{a \in A_i} \sigma(a)$, $a_j = \max_{a \in A_j} \sigma(a)$; we may assume that $a_i \le a_j$. Then for every $b \in B_j$ and $a \in A_i$ we have $\sigma(b) > a_j \ge a_i \ge \sigma(a)$; hence $A_i \cap B_j = \varnothing$. This contradiction shows that our claim is valid.

Next, let us estimate the total number of the numberings of ith type. The elements of $A_i \cup B_i$ can be ordered in $(a_i + b_i)!$ ways, and there are exactly $a_i!b_i!$ appropriate ones. Moreover, for each such order, there exists the same number of the numberings realizing this order. Hence the number of the numberings of ith type is exactly $m! \cdot \frac{a_i!b_i!}{(a_i+b_i)!} = m! \cdot \frac{1}{C^{a_i}_{a_i+b_i}}$.

Finally, the total number of the numberings is at least the sum of all the numbers found above, i.e.

$$\sum_{i=1}^{n} \frac{m!}{C^{a_i}_{a_i+b_i}} \le m!,$$

QED.

4.8. *Prove that there exist positive integers $a_1, \dots, a_n, b_1, \dots, b_n$ such that (i) $\sum_{i=1}^{n} \frac{1}{C^{a_i}_{a_i+b_i}} < \frac{1}{10^{100}}$, but (ii) there are no sets $A_1, A_2, \dots, A_n, B_1, \dots, B_n$ satisfying Katona's conditions such that $|A_i| = a_i$, $|B_i| = b_i$.*

Define $a_1 = 1$, $a_2 = 10^{101}$, $a_3 = 10^{101}$ and $b_1 = 10^{101}$, $b_2 = 1$, $b_3 = 1$. Then each of the sets B_2 and B_3 should have a common element with A_1. All these three sets are of cardinality 1, so they should coincide. Hence, since $B_2 \cap A_2 = \varnothing$, we also have $B_3 \cap A_2 = \varnothing$ which is impossible.

5.1. *Let $\mathcal{F}$ be a family of sets. Suppose that $|A \cap B| = 1$ for every distinct $A, B \in \mathcal{F}$ (in particular, this condition implies that $\mathcal{F}$ is a family of degree 1). Suppose, in addition, that among any four sets in $\mathcal{F}$, there exists three sets having a nonempty intersection. Prove that one can delete one of the sets from $\mathcal{F}$ so that all the remaining sets will have a nonempty intersection.*

Consider three sets with a nonempty intersection, say $x \in A \cap B \cap C$. Suppose that there exist two different sets D and E not containing x. Consider the four sets A, B, D, E. Some three of them have a common element y ($y \neq x$ since $x \notin D, E$). Then y belongs to at least one of the sets A and B; without loss of generality, $y \in A$. Next, A and B may have at most one element in their intersection, and $x \in A \cap B$; therefore $y \notin B$. Analogously, $y \notin C$. Then $y \in D$ and $y \in E$, in other case there is no three sets among A, B, D, E covering y.

Consider the four sets B, C, D, E. Among any three of them there are either both B and C, or both D and E. The unique common element of B and C is x, and it does not belong to D or E. Analogously, the unique common element of D and E is y, and it does not belong to B or C. Thus, the four sets B, C, D, E have no element belonging to three of them. This contradiction proves that there exists at most one set in $\mathcal{F}$ not containing x.

5.2. *Let $\mathcal{F}$ be a family of sets. Suppose that $\mathcal{F}$ contains at least 17 sets, and for every distinct $A, B \in \mathcal{F}$ we have $|A \cap B| = 1$. Moreover, suppose that among any 5 sets in $\mathcal{F}$, there are three sets with a nonempty intersection. Prove that there exists a 2-transversal for $\mathcal{F}$.*

Assume that for every four sets in $\mathcal{F}$, there are three with a nonempty intersection. Then, by the previous problem, some element x belongs toall the sets on $\mathcal{F}$ except one. Taking any element y from this exceptional set, we obtain a 2-transversal $\{x, y\}$ of $\mathcal{F}$.

Otherwise, there exist four sets A, B, C, D such that no three of them share a common element. Adding any other set $E \in \mathcal{F}$, we obtain a 5-tuple of sets three of which have a common element; these sets should be E together with some pair of the original four sets, and their common element is a (unique!) common element for this pair. There are 13 ways to choose $E \in \mathcal{F}$ and 6 ways to choose a pair of the original sets. Hence, some pair appears for three choices of E, and its common element x belongs to five sets in $\mathcal{F}$. Now we denote these five sets by X_1, X_2, X_3, X_4, X_5.

Consider now all the sets in $\mathcal{F}$ not containing x. If there are at most two of them, then their common element y together with x form a desired 2-transversal for $\mathcal{F}$. Next, if every three of these sets have a common element, then it is easy to see that this element y should be the same for all the triples, and $\{x, y\}$ is a 2-transversal for $\mathcal{F}$ again.

In the only case remaining, there exist three sets Y, Z, T not containing x and with no common element. Denote their pairwise common elements by a_1, a_2, a_3. Note that a_i cannot belong to two of the sets $X_1, \dots, X_5$ since otherwise these two sets have two common elements: x and a_i. Hence, among these five sets there are two (say X_4 and X_5) containing none of a_i's. Finally, consider five sets X_4, X_5, Y, Z, T; no three of them have a common element. A contradiction.

Firstly, we will present the solutions of problems 5.10 and 5.12–5.14. We will use them further as lemmas.

In the sequel, by $\mathcal{G}$ we always denote a (d, t)-exceptional family.

One may easily check that $X_A \not\subseteq X_B$ for arbitrary distinct $A, B \in \mathcal{G}$.

We need the following easy lemmas.

Lemma 3. *Let S be some set, $|S| = s$. Consider the subfamily $\mathcal{F} = \{A : A \in \mathcal{G}, S \subseteq X_A\}$. Then $\mathcal{F}$ is a $(d, t - s)$-exceptional family.*

Proof. If $S \subseteq X_A$ then $S \cap A = \varnothing$. Thus for all distinct $A, B \in \mathcal{F}$ we have $A \cap (X_B \setminus S) = A \cap X_B \neq \varnothing$. Hence, the set $X_A \setminus S$ has a common element with each set in $\mathcal{F}$ except A. Moreover, we have $|X_A \setminus S| = t - s$. That means exactly that $\mathcal{F}$ is $(d; t - s)$-exceptional. □

Lemma 4. *Let S be some set, $|S| = s$. Consider the family $\mathcal{F} = \{A \setminus S : A \in \mathcal{G}, S \subset A\} \subset \mathcal{G}$. Then $\mathcal{F}$ is a $(d - s, t)$-exceptional family.*

Proof. For every distinct $A, B \in \mathcal{G}$, we have $|(A \setminus S) \cap (B \setminus S)| = |A \cap B| - s \le d - s$. Also, since $X_A \cap S = \varnothing$ we have $(B \setminus S) \cap X_A = B \cap X_A \neq \varnothing$. Finally, $(A \setminus S) \cap X_A \subseteq A \cap X_A = \varnothing$. □

5.10. *Prove that $b(d;t) \ge a(d;t)$.*

It is sufficient to prove that for any integer n, $a(d;t) > n$ implies $b(d;t) > n$.

If $a(d;t) > n$, then there exist a family $\mathcal{G}$ of degree d such that any n sets in $\mathcal{G}$ have a t-transversal, but the whole family does not. Consider the minimal (with respect of number of sets) subfamily $\mathcal{F} \subset \mathcal{G}$ having no t-transversal. Obviously, $|\mathcal{F}| > n$. Since $\mathcal{F}$ is minimal, any subfamily of $\mathcal{F}$ has a t-transversal, thus $\mathcal{F}$ is (d,t)-exceptional. Then $b(d;t) > n$.

Remark. In the sequel, almost all the upper bounds for $a(d;t)$ are in fact the upper bounds for $b(d;t)$. In this case we just omit mentioning that $b(d;t) \ge a(d;t)$.

5.12. *a) Let $t = 2$. Prove that $h(x) \le d + 2$ for arbitrary x.*

b) Prove that $h(x) \le b(d; t-1)$ for arbitrary d, t, x.

Although item b) is a generalization of item a), we present a separate proof of item a) to clarify the idea.

a) Suppose $h(x) \ge d + 3$, i.e. there exist $A_1, \dots, A_{d+3} \in \mathcal{G}$ such that $X_{A_i} = \{x, x_i\}$ for some elements x_i (obviously, the elements x_i are distinct). Then $\{x_1, x_2, \dots, x_{d+1}\} \subset A_{d+2}$ and $\{x_1, x_2, \dots, x_{d+1}\} \subset A_{d+3}$ which contradicts the condition $|A_{d+2} \cap A_{d+3}| > d$.

b) Applying Lemma 3 for the case $S = \{x\}$, we obtain that the family $\mathcal{F} = \{A \in \mathcal{G} : x \in X_A\}$ is $(d, t-1)$-exceptional, thus $h(x) = |\mathcal{F}| \le b(d; t-1)$.

5.13. *Prove that $g(x) \le b(d-1;t)$ for every x.*

Applying Lemma 4 to the set $S = \{x\}$, we obtain that $\mathcal{F} = \{A \setminus \{x\} : A \in \mathcal{G}, x \in A\}$ is $(d-1,t)$-exceptional, i.e. $g(x) = |\mathcal{F}| \le b(d-1;t)$.

5.14. *Assume that $t = 2$ and $h(x) \le d$. Prove that $|\mathcal{G}| \le g(x) + h(x) + b(d - h(x); 2)$.*

Let $B_1, \dots, B_h \in \mathcal{G}$ be all the sets in $\mathcal{G}$ that contain x (then $h = h(x)$). Let $X_{B_i} = \{x, x_i\}$; again all x_i are distinct.

Let $A_1, \dots, A_k \in \mathcal{G}$ be all the sets such that $x \notin A_i$ and $x \notin X_{A_i}$. Since $k = |\mathcal{G}| - g(x) - h(x)$, it is sufficient to prove that $k \le b(d - h(x); 2)$.

Since $A_i \neq B_j$ for all i and j, we get $\{x, x_j\} \cap A_i \neq \varnothing$. Keeping in mind that $x \notin A_i$ we get $x_j \in A_i$. Denote $Y = \{x_1, \dots, x_h\}$; we have proved that $Y \subset A_i$ for all $i = 1, \dots, k$. Now, applying Lemma 4 for $S = Y$ we get $k \le b(d-h;2)$.

Remark. Both the statement and the proof remain valid for the case $h(x) > d$; note that we have $b(d-h;t) = 1$ for $d < h$.

Now, we firstly prove the serial bounds, although using sometimes the sharp partial bounds obtained further.

Since $a(d;t) \le b(d;t)$ (see problem 5.10) it is sufficient to prove the required upper bounds for $b(d;t)$ instead of $a(d;t)$. Thus in such cases we always consider some $(d;t)$-exceptional family $\mathcal{G}$ and prove some upper bounds for $|\mathcal{G}|$.

5.8. *a) Prove that $a(d;t) \ge C^t_{d+t+1}$.*

*b)** Do there exist the values of d and t such that the inequality in a) is strict?*

a) Consider a set B, $|B| = d + t + 1$. Let $\mathcal{G}$ be the family of all $(d+1)$-element subsets of B. Then for any distinct $A_1, A_2 \in \mathcal{G}$ we have $|A_1 \cap A_2| \le d$. Also, for any $A \in \mathcal{G}$ we put $X_A = B \setminus A$. Thus $|X_A| = t$ and $A_1 \cap X_A \neq \varnothing$ for any distinct $A, A_1 \in \mathcal{G}$. So, $\mathcal{G}$ is (d,t)-exceptional.

b) The solution of this item is still not known.

5.3. *Prove that $a(d;1) = d + 2$.*

Suppose that a $(d;1)$-exceptional family $\mathcal{G}$ contains at least $d+3$ sets, say $\{A_1, A_2, \dots, A_{d+3}\} \subseteq \mathcal{G}$. Let $X_{A_i} = \{x_i\}$, all x_i are distinct for $i = 1, \dots, d+3$. By the definition of a $(d;t)$-exceptional family we have $x_i \in A_1$ for $i = 3, \dots, d+3$. Analogously $x_i \in A_2$ for $i = 3, \dots, d+3$. Thus $|A_1 \cap A_2| \ge |\{x_3, \dots, x_{d+3}\}| = d + 1$. A contradiction.

Remark. Further we will also use an obvious proposition $a(0;t) = b(0;t) = t + 1$.

For the sake of brevity we introduce the following notation.

Definition 1. *Let $G_x = \{A \in \mathcal{G} : x \in A\}$ and $H_x = \{A \in \mathcal{G} : x \in X_A\}$ for arbitrary x.*

For the induction step in the next problem, we need the following sharpening of the statement of problem 5.12a).

Lemma 5. *Assume that $t = 2$ and $|\mathcal{G}| > b(d-1; 2) + d + 3$. Then $h(x) \le d$ for all x.*

Proof. Assume, to the contrary, that there exist $d+1$ distinct sets $A_1, \dots, A_{d+1}$ such that $x \in X_{A_i}$. Let $X_{A_i} = \{x, x_i\}$ (again all x_i are distinct). Applying problem 5.13 we get $g(x) \le b(d-1; 2)$, so there are at least $d+3$ sets in $\mathcal{G} \setminus G_x$ and at least two of these sets are distinct from $A_1, \dots, A_{d+1}$. Denote those two sets by B and C. Since $x \notin B$ but $X_{A_i} \cap B \ne \varnothing$ we conclude $\{x_1, \dots, x_{d+1}\} \subseteq B$. Analogously $\{x_1, \dots, x_{d+1}\} \subseteq C$. Thus $|B \cap C| \ge d+1$. A contradiction. □

Remark. In fact, this proof is a repetition of the solution of problem 5.14 (or its version for $h > d$).

5.5. *a) a) Prove that $a(d; 2) \ge C^2_{d+3}$.*
b) Prove that $a(d; 2) \le 2d^2 + 3$ for all $d \ge 2$.

a) Follows from problem 5.8a).

б) We use the induction on $d \ge 2$ to prove that $b(d; 2) \le 2d^2 + 3$. The base case will follow from the step.

Assume first that $b(d; 2) \ge b(d-1; 2) + d + 3$. Then for all x we have $h(x) \le d$ by Lemma 5. Let $b = |\mathcal{G}|$.

Now, for every element x denote by $n(x)$ the number of (ordered) triples of distinct sets (A, B, C) from $\mathcal{G}$ such that $x \in A$, $x \in B$, and $x \in X_C$. We will bound the sum σ of all the numbers $n(x)$ in two different ways.

On one hand, there exist $b(b-1)$ pairs of different sets (A, B). For each such pair, there are not more than d ways to choose an element $x \in A \cap B$, and for every such element there exist $h(x) \le d$ possibilities for C. Thus, we get $\sigma = \sum_x n(x) \le b(b-1) \cdot d^2$.

On the other hand, consider an arbitrary set C; let $X_C = \{x, y\}$. Each of the other $b-1$ sets in $\mathcal{G}$ contains either x or y. Denote by s_x and s_y the number of sets containing x and y, respectively. Then $s_x + s_y \ge b - 1$. Furthermore, there exist $s_x(s_x - 1)$ pairs of the sets intersecting by x, as well as $s_y(s_y - 1)$ pairs intersecting by y. In all, the triples with this fixed C are counted exactly $s(C) = s_x(s_x - 1) + s_y(s_y - 1)$ times. From $s_x + s_y \ge b - 1$ one easily gets $s(C) \ge \frac{(b-1)(b-3)}{2}$ (the estimate is sharp for $s_x = s_y = \frac{b-1}{2}$). Thus, we get $\sigma \ge b\frac{(b-1)(b-3)}{2}$.

Finally we get $d^2 b(b-1) \ge \sigma \ge \frac{1}{2} b(b-1)(b-3)$, hence $2d^2 \le b - 3$, QED.

The only remaining case is $b(d; 2) \le b(d-1; 2) + d + 2$. If $d \ge 3$, then by the induction hypothesis we have $b(d; 2) \le (2(d-1)^2 + 3) + d + 2 = 2d^2 - 3d + 5 \le 2d^2 + 3$, as desired. Finally, if $d = 2$, then we use the problem 5.4a) to obtain $b(1; 2) = 6$, therefore $b(2; 2) \le 6 + 2 + 2 = 10$ which is even stronger than we need.

Remark. Note that the estimates in main part of the solution admit some improvements. Thus, the upper bound for σ is sharp only if $h(x) = d$ for all x. On the other hand, one may see from problem 5.14 that under this restriction $g(x)$ is much greater than $\frac{b-1}{2}$, and the lower bound is far from being sharp. The method of obtaining sharper bounds is shown in the proof of the Lemma 7 before the computation of $b(1; 4)$ and applied after that.

5.7. *a) Prove that $a(1; t) \ge C^2_{t+2}$.*
b) Prove that $a(1; t) \le t^2 + 1$ for all $t \ge 3$.

a) Follows from problem 5.8a).

б) The solution is literally the same as that for 3.4a): the only property of the lines which is used there (but not in 3.4b)!) is exactly that they form a family of degree 1.

5.11. *Prove that the number $b(d; t)$ is finite for all d and t.*

We prove the estimate $b(d; t) \le t^{d+1} + (t^{d-1} + t^{d-2} + \dots + t^0)$ by the induction on $d \ge 1$. The base case $d = 1$ is proved in 5.7б).

For the induction step, let $d \ge 2$. Consider an arbitrary set $A \in \mathcal{G}$; let $X_A = \{x_1, \dots, x_t\}$. Then each of the other sets in $\mathcal{G}$ contains at least one of the elements $x_1, \dots, x_t$. Next, the number of sets in $\mathcal{G}$ containing x_i is at most $b(d-1;t)$ by Lemma 4. Therefore $|\mathcal{G}| \le 1 + tb(d-1;t) = 1 + t^{d+2} + (t^d + t^{d-1} + \cdots + t)$, as desired.

Remark. Instead of 5.7б), one may use a trivial bound $b(0;t) \le t+1$ as the base case. This way, one obtains a bit weaker bound, namely $b(d;t) \le t^{d+1} + t^d + \cdots + 1$.

5.9. *a) Prove that the number $a(d;t)$ is finite for every pair (d,t).*
b) Try to obtain a good[1] *bound for the number $a(d;t)$.*

Item a) is an immediate consequence of 5.10 and 5.11. From the solutions of the same problems one can obtain $a(d;t) \le t^{d+1} + (t^{d-1} + t^{d-2} + \cdots + t^0)$ for all $d \ge 1$.

Finally, we proceed to sharp (although particular) results from problems 5.4 and 5.6. Here we do not refer to the numbers of problems but just write down the equalities to be proved.

The lower bounds in all cases follow directly from problem 5.8a). For the upper bounds we assume again that a $(d;t)$-critical family $\mathcal{G}$ is chosen and we bound the cardinality of the family.

$\boldsymbol{b(1;2) = 6}$. Assume that $|\mathcal{G}| \ge 7$. Consider an arbitrary $C \in \mathcal{G}$; set $X_C = \{x, y\}$. Then each of the remaining sets in $\mathcal{G}$ contains either x or y; hence at least three of the sets contain the same element of X_C, say $x \in A_1, A_2, A_3$. Notice that $g(x) \le 3$ by problem 5.13; so the remaining sets in $\mathcal{G}$ do not contain x. By our assumption, $\mathcal{G}$ contains at least 4 sets C_1, C_2, C_3, C_4 distinct from A_i.

Consider an arbitrary index $k = 1,2,3,4$. Set $X_{C_k} = \{x_1, x_2\}$. There must be two sets among A_i containing the same element x_j. Since their intersection consists of x only (because $\mathcal{G}$ is a family of degree 1!) it follows that $x = x_j \in X_{C_k}$. Thus $x \in C_k$ for each $k = 1,2,3,4$. Then by Lemma 3 applied to $S = \{x\}$ the family $\{C_1, C_2, C_3, C_4\}$ is $(1;1)$-exceptional, and by problem 5.3 it has at most 3 elements. A contradiction.

Remark. Analyzing this solution one can capture the following lemma useful for the sequel.

Lemma 6. *Let $\mathcal{G}$ be an exceptional $(1,t)$-family such that $g(x) \ge t+1$ for some x. Then for each $B \in \mathcal{G}$, we have either $x \in B$ or $x \in X_B$. Moreover, in this case we have that $g(x) = t+1$ and the family $\mathcal{F} = \mathcal{G} \setminus G_x$ is $(1,t-1)$-exceptional.*

Proof. Assume that $g(x) \ge t+1$; we have $G_x \supseteq \{A_1, \dots, A_{t+1}\}$.

Assume that $x \notin B$ and $x \notin X_B$ for some $B \in \mathcal{G}$. Each of the sets $A_1, \dots, A_{t+1}$ contains at least one element of X_B; since $|X_B| = t$ it follows that two of these sets contain the same element (say, $y \in A_1 \cap A_2 \cap X_B$). But then $A_1 \cap A_2$ contains (distinct!) elements x and y, which is impossible. Thus the required set B does not exist.

Further, $g(x) \le t+1$ by problem 5.13, so that $g(x) = t+1$. The second assertion now follows from Lemma 3. □

$\boldsymbol{b(1;3) = 10}$. Assume that $|\mathcal{G}| \ge 11$. Consider an arbitrary $C \in \mathcal{G}$; set $X_C = \{x, y, z\}$. Then each of the remaining sets in $\mathcal{G}$ contains at least one of the elements x, y, or z, thus $g(x)+g(y)+g(z) \ge 10$. Hence at least one of the summands is greater than 3, say, $g(x) \ge 4$. Then by Lemma 6 we obtain that $\mathcal{F} = \mathcal{G} \setminus G_x$ is $(1;2)$-exceptional, and $|\mathcal{F}| \ge 7$. This contradicts to $b(1;2) = 6$.

Remark. The previous two statements essentially repeat the solution of problem 3.4a). We present them here to make the appreciation of the next point more convenient.

For the sequel we improve somehow the main approach of problem 5.5. That is, we slightly modify the definition of the sum σ, so that it allows to obtain a bit better estimates. The meaning of the following definition is clear from the lemma afterwards.

Definition 2. *For each element x belonging to at least one set of the form X_A, the* price *of x is $p(x) = \dfrac{g(x)(g(x)-1)}{h(x)}$. For each $C \in \mathcal{G}$, define its* price *by the formula $P(C) = \sum_{x \in X_C} p(x)$.*

Lemma 7. *Let $\mathcal{G}$ be a $(d;t)$-critical family of cardinality b. Then there exists a set $C \in \mathcal{G}$ such that $P(C) \le d(b-1)$.*

[1] as good as possible…

Proof. Again, for each element x denote by $n(x)$ the number of ordered triples of distinct sets (A, B, C) from the family $\mathcal{G}$ such that $x \in A$, $x \in B$, and $x \in X_C$. But now we are going to estimate a different sum, that is

$$\Sigma = \sum_x \frac{n(x)}{h(x)}$$

(naturally, the sum is over all the elements appearing in the sets of the form X_A).

On one hand, there exist $b(b-1)$ pairs of distinct sets (A, B). For each such pair there is no more than d possibilities to choose an appropriate element x (because $|A \cap B| \leq d$). Further, any element $x \in A \cap B$ contributes exactly 1 to the sum Σ; indeed, there exist $h(x)$ possibilities for the set C, and each of these possibilities is counted with the "weight" $1/h(x)$. Thus we get

$$\Sigma = \sum_{(A,B):\ A \neq B} |A \cap B| \leq d \cdot b(b-1).$$

On the other hand, we consider an arbitrary set C, and we show that its contribution to Σ is equal to $P(C)$. Indeed, for each $x \in X_C$, we count exactly $g(x)(g(x)-1)$ pairs in the number $n(x)$. Taking into account the "weight" $1/h(x)$, we obtain that the "contribution of the pair x, C" equals $p(x)$. Therefore the contribution of the set C equals $P(C)$, as required.

So $db(b-1) \geq \Sigma = \sum_{C \in \mathcal{G}} P(C)$. By the pigeonhole principle, one of the b summands from the right-hand side is not greater than $d(b-1)$. □

$b(1;4) = 15$. Assume that $|\mathcal{G}| \geq 16$; removing several sets from $\mathcal{G}$ we may also assume that $b = |\mathcal{G}| = 16$. Consider an arbitrary $C \in \mathcal{G}$; set $X_C = \{x_1, x_2, x_3, x_4\}$. Analogously to the previous case, we get $\sum_i g(x_i) \geq 15$, and $g(x_i) \leq 4$ (otherwise, using Lemma 6, we get a contradiction with $b(1;3) = 10$).

So $\sum_i g(x_i) = \sum_i |G_{x_i}| \leq 16$, but each set $A \in \mathcal{G}$ distinct from C must be contained in at least one of G_{x_i}. This implies that either all the families G_{x_i} are disjoint or only two of them intersect each other by a single set. In any case we may assume that G_{x_1}, G_{x_2}, and G_{x_3} are disjoint and contain 4 sets each.

Proposition. For each $A \in \mathcal{G}$, which is distinct from C, the set X_A contains neither of the elements x_1, x_2, x_3.

Proof. Assume that $x_1 \in X_A$; then clearly $x_1 \notin A$. Notice that one of the two elements x_2, x_3 also does not belong to A, because $G_{x_2} \cap G_{x_3} = \varnothing$; we may assume that $x_2 \notin A$. Suppose now that $x_2 \notin X_A$. Denote by B_1, B_2, B_3, B_4 the four sets containing x_2; none of them contains x_1. The intersection of any two of them is $\{x_2\}$; hence all the sets $B_i \cap X_A$ are disjoint. Thus X_A consists of these four intersections (because $|X_A| = 4$). But they do not contain x_1. This contradicts to the assumption that $x_1 \in X_A$.

We have proved that $x_1, x_2 \in X_A$. Now assume that $x_3 \notin X_A$. The family G_{x_3} contains at least three sets distinct from A. They must intersect with X_A by distinct elements, which are also distinct from x_1, x_2. This is again impossible because $|X_A| = 4$. Thus $X_A = \{x_1, x_2, x_3, y\}$ for some $y \neq x_4$.

Finally, consider two sets $B, B' \in \mathcal{G}$ which do not belong to $G_{x_1} \cup G_{x_2} \cup G_{x_3}$ and are distinct from A and C (such sets exist because $|G_{x_1}| + |G_{x_1}| + |G_{x_1}| + 2 = 14 = |\mathcal{G}| - 2$). Then $\varnothing \neq B \cap X_C$, therefore $x_4 \in B$. Analogously, $y \in B$, and also $x_4, y \in B'$. Thus $|B \cap B'| \geq 2$, which is impossible. □

So we know that $h(x_1) = h(x_2) = h(x_3) = 1$ and $g(x_1) = g(x_2) = g(x_3) = 4$. Hence $P(C) \geq 3 \cdot \frac{12}{1} + p(x_4) > 36$. On the other hand, by Lemma 7 there exists $C \in \mathcal{G}$ such that $P(C) \leq db = 30 < 36$. This contradiction concludes the proof.

$b(2;2) = 10$. Assume that $|\mathcal{G}| = 11$. It follows from Lemma 5 that $h(x) \leq 2$ for each x, and by problem 5.13 we have $g(x) \leq 6$. Then by problem 5.14, we get that $g(x) = 6$ whenever $h(x) = 2$. Thus if $g(x)$ attains one of the values 4, 5 or 6, then $p(x)$ cannot be less than 12, 20 or $\frac{30}{2} = 15$, respectively.

By Lemma 7 there exist $C \in \mathcal{G}$ such that $P(C) \leq 2 \cdot 10 = 20$. Let $X_C = \{x, y\}$, where $g(x) \geq g(y)$. Notice that $g(x) + g(y) \geq |G_x \cup G_y| = 10$, so that $g(x) \geq 5$, $g(y) \geq 4$. Then $P(C) \geq 15 + 12 = 27$. A contradiction.

_b_(3; 2) = 15. Assume that $|\mathcal{G}| = 16$. Arguing analogously to the previous solution we get $h(x) \le 3$, $g(x) \le 10$, and for $h(x) = 1, 2, 3$ we have $g(x) \ge 5, 8, 10$, respectively.

Further, using Lemma 7 we find $C \in \mathcal{G}$ such that $P(C) \le 3 \cdot 15 = 45$. Set $X_C = \{x, y\}$, where $g(x) \ge g(y)$. Then $g(x) \ge 8$, therefore $p(x) \ge 28$; also $p(y) \ge 20$, because $g(y) \ge 5$. Thus $P(C) \ge 48$. A contradiction.

_b_(4; 2) = 21. Arguing analogously to the previous solutions, we find $C \in \mathcal{G}$ such that $P(C) \le 84$. If $X_C = \{x, y\}$ and $g(x) \ge g(y)$, then we get an analogous contradiction in every case except $g(y) = 6$; in this remaining case we necessarily have $g(x) = 15$ and $h(x) = 4$ (otherwise $P(C) > 84$ again).

Consider the last case separately. Since $h(x) = 4$, there exist sets C_1, C_2, C_3, C_4 such that $h(C_i) = \{x, y_i\}$. Then there exist $22 - g(x) - h(x) = 3$ sets B_1, B_2, B_3 such that $x \notin B_i$ and $x \notin X_{B_i}$. By the properties of the sets X_{C_j} it follows that $\{y_1, y_2, y_3, y_4\} \subseteq B_i$. Thus any two sets B_i must intersect precisely by the elements y_1, y_2, y_3, y_4.

Finally, for each $C \in \mathcal{G}$, which is distinct from B_i, the set X_C intersects all B_i; thus one its element is contained in two sets B_i, B_j. Thus one of the elements y_1, y_2, y_3, y_4 is contained in X_C. This implies that $\sum_i h(y_i) \ge 22 - 3 = 19$; but then $h(y_i) \ge 5$ for some i. As we have noticed above, this is impossible.

Remark. Developing these approaches the authors have proved that $a(d; 2) = b(d; 2) = C^2_{d+2}$ for each $d \le 7$.