Part A.

Solutions.

1. This immediately follows from calculation of angles.

2. This item clearly follows from the previous item.

3. It is clearly follows from the fact that the angle between bisector of angle $\angle A$ and line B_1C_1 is right.

4. This item follows from two previous and the properties of the inscribed angle.

5. Let E be a midpoint of segment A_1C_1. Let X be an intersection point of B_1E with incircle different of B_1. Let's calculate degree of point E with respect to the circumcircle of BA_1IC_1. We get $BE \cdot EI = EA_1 \cdot EC_1$. Now write the equation on degree wrt the incircle $EA_1 \cdot EC_1 = EX \cdot EB_1$. So by $BE \cdot EI = EX \cdot EB_1$ we obtain that quadrilateral $BXIB_1$ is inscribed. $IX = IB_1$ as radii. So angles $\angle B_1BI$ and $\angle IBX$ are equal. Let Y be an intersection point of BB_1 and the incircle different of B_1. From the above we obtain that $XY \parallel A_1C_1$. Hence lines B_1X and B_1Y are isogonaly conjugated in angle $\angle A_1B_1C_1$. Let's project cross-ratio of points A_1, C_1, E and infinity point of line A_1C_1 from point X into the incircle. Since cross-ratio of the initial quadruple equals -1 cross-ratio of images is equals -1 also. So we get that $(A_1, C_1, B_1, Y) = -1$ for any concyclic quadrilateral in which tangents through two of their vertices intersect on diagonal passing through two other vertices. As you remember such quadrilateral are called harmonic.

6. It is easy to prove this fact using previous item and angles calculation.

7. At first we draw tangents from point B wrt the incircle. So we obtain points A_1 and C_1. Now construct midpoint of A_1C_1 point E. From previous item we know that the circumcircle of triangle A_1IC_1 divide segment B_1E in ratio $2:1$. So if we make homothety with center E and coefficient 3 then the image of the circumcircle of triangle A_1IC_1 pass through B_1. So if we intersect this image with the incircle we obtain two symmetrical solutions of point B_1. Now it is easy to construct vertices A_1 and C_1.

8. From solution of previous item clearly follows that IE must be at most than half of radius of the incircle. From this fact it easy to prove that $\angle A_1IC_1$ is at least $120°$. So angle $\angle B$ which is equal $180° - \angle A_1IC_1$ is at most $60°$.

Часть B.

Решения.

Пусть вторая точка пересечения AA_1 со вписанной окружностью — точка Q, а вторая точка пересечения CC_1 со вписанной окружностью — P. Теперь пусть K — бегающая точка по вписанной окружности. Обозначим точку пересечения KP с BC через P', а KQ с AB через Q'.

Эту часть можно не решать. Факты из нее не используются в дальнейшем.

1. Since $\angle PGQ = 90^\circ + \frac{\angle B}{2}$, we obtain that the sum of arcs PQ and A_1B_1 is equal to $180^\circ + \angle B$. Since arc A_1C_1 is equal to $180^\circ - \angle B$, arc PQ is equal to $2\angle B$. Thus, if the line passing through Q and parallel to BA meets secondary the incircle in point T, then $\angle QTP = \angle B$. Therefore $TP \parallel BC$.

2. Let point P' move on BC with constant velocity. Then Q' moves in such way that the cross-ratios of different points P' and Q' are equal. Therefore the correspondence between P' and Q' is projective. But by previous item P' and Q' are infinity points of the respective lines simultaneously. Thus this correspondence is affine, i.e. point Q also moves with constant velocity. Therefore the locus of midpoints is a line. Considering the cases $K = A_1$ and $K = C_1$ we obtain that this line is the Gauss line of quadrilateral AC_1A_1C.

3. Since the locus from previous item is the Gauss line of A_1C_1AC, it contains the midpoints of third "diagonal". Let R' be the midpoint of RB. Find points P' and Q' on AB and BC respectively such that R' is the midpoint of $P'Q'$. Then $P'BQ'R$ is a parallelogram i.e. P', Q' coincide with P, Q.

4. Note that the projection of a circle to the same circle from a point not lying on this circle conserves the cross-ratios. In fact, let the center of projection lie outside the circle. Then there exists a projective map conserving the circle and transforming the projection center to an infinity point. The projection from such point coincides with the reflection in some diameter and so conserves the cross-ratios. If the projecton center lies inside the circle we can transform this point to the center of the circle. In this case the projection coincides with the reflection in the center and also conserves the cross-ratios. Now note that quadrilateral $B_1C_1A_1P$ is harmonic because the tangents in points B_1 and A_1 meet on diagonal C_1P. Consider the projection of this quadrilateral from point $R$to the same incircle. It interchanges points A_1 and C_1 and fixes B_1. Thus the image of P is the fourth harmonic point for B_1, A_1, C_1, i.e. point Q. Therefore P, Q and R are collinear.

5. Points R, B_1, A, C are harmonic because the Gergonne triangle is cevian. Project these points from the infinity point of AB to BC. Let the images of R, A, C, B_1 be P_1, B, C, B_1' respectively. Project these four points from B_1 to AB. This projection transforms B_1' to the infinity point, fixes B and transforms C to A. Since four points are harmonic it transforms P_1 to the midpoint, i.e. $P_1 = P_2$. Now use the assertion of problem 2.

6. Note that if $P' = A$ then $Q' = A_1$, and if $P' = C_1$ then $Q' = C$. Therefore when one of points is the midpoint of the corresponding segment the second one also is the midpoint.

7. Consider the projection of line BC from P to the incircle. It transforms P', A_1, C and the infinity point to K, A_1, C_1 and T respectively. Since the cross-ratio of these points is equal to -1, the diagonal TK passes through the common point of tangents in A_1 and C_1.

8. Arcs TC_1 and C_1Q are equal because the tangent in C_1 is parallel to QT. Arcs C_1Q and A_1F' are equal because $C_1F' \parallel QA_1$. Thus arcs C_1T and $F'A_1$ are equal which yields the symmetry wrt the bisector of $\angle B$.

9. See the solution of problem 5 from part A. Inversing its reasoning and using two previous items we obtain the assertion of the problem.

10. Consider the homothety with center K transforming P to P'. It transforms T to B and thus the image of Q is Q'. Since $Q'E$ is parallel to QG as medial line, we have thatit transforms G to M. Since PG passes through the midpoint of arc QT, PQ is the bisector of triangle PQT. Then QA_1 also is the bisector. Therefore G is the incenter of triangle QTP. Thus its image M is the incenter of triangle $Q'BP'$.

Part C.

Solutions.

1. Let lines BC' and AC meet in point R. For prove that B lies on $A'C'$, it is sufficient to prove that cross-ratios (A, C_0, B, C_1) and (A, B_0, R, B_1) are equal. By the Ceva theorem $\frac{AC_0}{C_0B} = \frac{AB_0}{B_0C} \cdot \frac{CA_0}{A_0B}$, $\frac{AC_1}{C_1B} = \frac{AB_1}{B_1C} \cdot \frac{CA_1}{A_1B}$. From this and an equality $(C, A_0, B, A_1) = (C, B_0, R, B_1)$ (projection from C') we obtain
$$\frac{AC_0}{C_0B} \cdot \frac{C_1B}{AC_1} = \frac{AB_0}{B_0C} \cdot \frac{CA_0}{A_0B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \frac{AB_0}{B_0C} \cdot \frac{CB_0}{B_0R} \cdot \frac{B_1R}{B_1C} \cdot \frac{CB_1}{B_1A} = \frac{AB_0}{B_0R} \cdot \frac{B_1R}{AB_1}$$
.

2. Let lines A_1C_1 and AC meet in point R. Project cross-ratio $(A, C, B_1, R) = -1$ to line B_1C_1 from point B'. This projection transforms C, A, R to A', X, C_1respectively and fixes B_1. Therefore $(B_1, C_1, A', X) = -1$, i.e. the polar of A' passes through X and the pole of B_1C_1 coinciding with A. Thus it is line AX which coincide with $B'C'$ by previous item.

3. **Lemma.** Let triangle $A_1B_1C_1$ be cevian for ABC. Points A', B' and C' on its sidelines are such that the vertices of triangle ABC lie on the sidelines of triangle $A'B'C'$. Let A_0 be the common point of B_1C_1 and BC. Points B_0 and C_0 are defined similarly. Then lines AA', BB', CC' and $A_0B_0C_0$ concur.

   Proof. Consider triangles $CA'A_0$ and $C'AC_0$. We have $CA' \cap C'A = B'$, $A'A_0 \cap AC_0 = C_1$, $A_0C \cap C'C_0 = A_1$, and these three points are collinear. By Desargues theorem lines CC', AA', A_0C_0 concur. Since points A_0, C_0, B_0 are collinear (clearly follows from Ceva and Menelaos theorems), we obtain the assertion of the problem.

   Now note that in our case one of considered triangles is medial. Thus the corresponding lines meet on the infinity point, i.e. they are parallel.

4. Use the Pappus theorem to points A, B_1, C and C_1, B', A_1, and after this to points A, B_1, C and C_B, B', A_B. (see problem 7)

5. Similarly to previous item use the Pappus theorem to points A, B_0, C and C_0, B', A_0, and after this to A, B_0, C and C_A, B', A_C. (see problem 7)

6. Use the lemma of problem 3 to the Gergonne triangle.

7. $\angle B = 180^\circ - \angle A - \angle C \Rightarrow \angle BA_1C_1 = \frac{180^\circ - \angle B}{2} = \frac{\angle A + \angle C}{2} \Rightarrow \angle(A_1B, A_1C_1) = \angle(A_1C, CI) + \angle(AI, AC_1) \Rightarrow \angle(AI, AC_1) = \angle(A_1B, A_1C_1) - \angle(A_1C, CI) = \angle(A_1C, A_1C_A) - \angle(CA_1, CC_A) = \angle(IC_A, CA_1) + \angle(A_1C, C_1C_A) = \angle(IC_A, C_AC_1)$. Therefore A, C_1, C_A, I are concyclic.

8. By previous item C_A is the projection of A to the bisector of angle B. Thus angle AB_0C_A as central angle in triangle ACC_A is twice greater than angle C of this triangle, i.e. it is equal to angle C of triangle ABC. Therefore $B_0C_A \perp BC$, and C_A lies on the medial line.

9. Since C_0C_A is parallel to BA_1, triangle $C_1C_0C_A$ is isosceles. Thus $C_0C_1 = C_0C_A$. Similarly $C_0C_1 = C_0C_B$.

10. The circle from previous item is orthogonal to the incirle because it passes through $C_1$and its center lies on the tangent to the incircle in this point. Since C_A and C_B are collinear with the incenter, they are symmetric wrt the incircle. This is also true for points A_B and A_C. Thus $IC_A \cdot IC_B = r^2 = IA_B \cdot IA_C$, where r is the inradius. Therefore these four points lie on the circle which is fixed under the inversion and so is orthogonal to the incircle.

11. By the properties of polars and radical axis all these lines coincides with a common chord of given circles.

12. Using the Pascal theorem to points A, B, B, D, C, C we obtain that the tangents in B and C meet on line NP. Similarly the tangents in A and D meet on this line. Conisder now the polar of point M. This line passes through the poles of BC and AD. By above it is line PN. Thus M is the pole of PN. Similarly for two remaining points.

13. Point O_B lie on $A'C'$ iff the polar of O_B wrt the incircle passes through the pole of $A'C'$ i.e. point B'. Since ω_B is orthogonal to the incicrle the polar of O_B wrt the incircle coincide with the polar of I wrt ω_B. But for quadrilateral $C_BC_AA_BA_C$ inscribed into ω_B четырехугольника I is a common point of the oppsite sidelines, and B' is a common point of the diagonals. Thus the polar of I passes through B'.

14. Since ω_B, ω_A and the circumcircle of triangle $C_AC_BC_1$ passe through points C_A and C_B, their centers are collinear.

15. By previous item O_BO_A is perpendicular to C_AC_B, which is the bisector of angle C. Then it is also perpendicular to the bisecor of angle C_0 of triangle $A_0B_0C_0$. Therefore the sidelines of $O_AO_BO_C$ are the external bisectors of $A_0B_0C_0$. This clearly yields the assertion of the problem.

16. The sidelines of these triangles are perpendicular to the corresponding bisectors of original triangle.

17. Project cross-ratio $(A,\, C,\, B_1,\, R) = -1$ from B' to O_AO_C. This projection transforms A, C and R to O_A, O_C and infinity point respectively. Thus it transforms B_1 to M_B.

18. Note that M_A, M_B, M_C, A_0, B_0, C_0 lie on the Euler circle of triangle $O_AO_BO_C$. Then the homothety center of triangles $A_1B_1C_1$ and $M_AM_BM_C$ is also the homothety center of their circumcircles, i.e. the Euler circle of original triangle (because $\omega_{M_AM_BM_C} = \omega_{A_0B_0C_0} =$ is the Euler circle of original triangle) and its incircle. Therefofre this homothety center coincide with the Feuerbach point. But by previous item line M_AA_1 passes also through point A'. Thus A_1A' passes through the Feuerbach point.

19. Since $A_0B_0C_0$ is the medial triangle of ABC and the orthotriangle of $O_AO_BO_C$, the Euler circles of these two triangles coincide. Thus points M_A, M_B, M_C lie on the Euler circle of original triangle. But triangles $A_1B_1C_1$ and $M_AM_BM_C$ are homothetic, so their circumcircles are also homothetic. It is sufficient to prove that the homothety center lie on one of these circles. Let AA' meet the incircle in point R. Since B' lies on the polar of A' wrt the incircle and on the diagonal A_1C_1 of quadrilateral $A_1B_1C_1R$, we obtain that B' is the common point of the diagonals of this quadrilateral. The same is true for CC'. Thus the homothety center lies on the incircle and the circles touche.

20. This problem is dual to item 18.

21. See the solution of the next item.

22. **Lemma** 1. $\angle(A_0F,\, FA_1) = (\angle(CA,\, CB) + \angle(BA,\, BC))/2$.

Proof. Prove that $\angle(A_0F,\, FH) = \angle(CA,\, CB) + \angle(BA,\, BC)$, where $H$is the foot of altitude from A to BC. Then the sought assertion will follow from Archimedes lemma. Note that $\angle(A_0F,\, FH) = \angle(A_0B_0,\, B_0H)$ because all these points lie on the Euler circle. Also since B_0 is the midpoint of the hypothenuse of right triangle AHC, we have $\angle(CA,\, CB) = \angle(CB_0,\, CH) = \angle(HC,\, HB_0)$. Thus $\angle(A_0B_0,\, B_0H) = \angle(A_0B_0,\, A_0H) + \angle(A_0H,\, HB_0) = \angle(BA,\, BC) + \angle(CA,\, CB)$, q.e.d.

Lemma 2. Points B_C, C_B, A_1, A_0 and F are concyclic.

Proof. Prove that $\angle(A_0C_B,\, C_BA_1) = \angle(A_0F,\, FA_1)$. Since C_BIA_1B is cyclic we have $\angle(C_BI,\, C_BA_1) = \angle(BI,\, BA_1) = \angle(BA,\, BC)/2$. Since $C_BA_0 = A_0C$, we obtain an equality $\angle(A_0C_B,\, C_BC) = \angle(CC_B,\, C_A0) = \angle(CA,\, CB)/2$. From this $\angle(A_0C_B,\, C_BA_1) = \angle(A_0C_B,\, C_BC) + \angle(C_BC,\, C_BA_1) = \frac{(\angle(CA, CB) + \angle(BA, BC))}{2} = \angle(A_0F,\, FA_1)$. Thus C_B lies on the circumcircle of $\triangle A_1A_0F$. Similarly B_C lies on this circle.

23. We proved that A_1F is the radical axis of the incircle and the circumcircle of triangle $C_BB_CA_1$. Thus it is sufficient to prove that A' lies on this line. Note that A' is the homothety center of triangles $C_0C_1C_B$ and $B_0B_CB_1$ because the corresponding sidelines of these triangles are parallel. Therefore $\frac{A'C_1}{A'B_C} = \frac{A'C_B}{A'B_1}$, and we obtain an equality of degrees: $A'C_1 \cdot A'B_1 = A'C_B \cdot A'B_C$.