Fractional iterations of functions.

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We denote $f^{(n)}(x) = f(f(\dots x)\dots)$ (n times). Functions $f^{(n)}$ are *iterations* of f. If f is invertible, then *integer iterations* exist and $f^{(-n)} = (f^{(-1)})^{(n)}$. It is easy to see that $(f^{(n)})^{(m)} = f^{(nm)}$ and $f^{(n)} \circ f^{(m)} = f^{(n+m)}$.

How to define *fractional iterations*? What does it mean: *functional root* $g(x) = f^{(1/2)}(x)$? It is natural to assume that $g^{(2)} = f$. After defining *functional roots* $f^{(1/n)}$ one can define rational powers $f^{(m/n)} = (f^{(1/n)})^{(m)}$ and try to define real iterations via limit. (By the way, it is interesting what is the sense of complex or p-adic iterations.)

This way meets some difficulties. Invertibility of functions already is not always clear. We shall start from investigation of usual iterations and functional roots from set-theoretic point of view and from some important partial cases.

1 Functional roots

1. a) Does there exist a function g such that $g^{(2)}(x) = \cos x$?

   b) Does there exist a function g such that $g^{(2)}(x) = \sin x$?

   c) The same questions for continuous function g.

   d) The same questions for function g with finitely many discontinuity points.

2. a) Does there exist a function g such that $g^{(3)}(x) = e^{-x}$?

   b) The same question for a function g with finitely many discontinuity points.

3. a) Does there exist a function $f : (-1, 1) \to (-1, 1)$ such that $f(f(x)) = -x$?

   b) The same question for function f with finitely many discontinuity points.

   c) Does there exist a function $f : [-1, 1] \to [-1, 1]$ with finitely many discontinuity points such that $f(f(x)) = -x$?

4. $*$ Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ with finitely many discontinuity points such that $f(f(x)) = g$ for some strictly decreasing function g

5. a) How many permutations on 4-element set are squares?

   b) Describe all permutations on 9-element set which are cubes.

2 Fractional iterations of some elementary functions

1. a) Let $f(x) = x+c$. What are fractional iterations of f? b) Let $f(x) = \alpha x$. What are fractional iterations of f?

   c) Let $f(x) = \alpha x + c$. What are fractional iterations of f?

   d) Let $f(x) = \alpha x^n$. What are fractional iterations of f?

2. Let $f(x) = x^2 - 2$. What are fractional iterations of f?

3. a) $z_0 \notin \mathbb{R}$, $f(z) = z^2 - 2$. Prove that $|f^{(n)}(z_0)| \to \infty$ if $n \to \infty$. Find the set of initial points z_0 such that $|f^{(n)}(z_0)| = 2$.

) Let P be a polynomial with integer coefficients with oldest coefficient equal 1. Let all complex roots P are in the unit disc $|z| \leq 1$. Prove that they are roots of unit.

) Let P be a polynomial with integer coefficients with oldest coefficient equal 1. Let all complex roots P are distinct and in the line seqment $[-1.99, +1.99]$. Prove that the set of such polynomials is finite.

3 Some limits

1. On the rectangular map M somebody put a map N of same landscape but smaller masshtab. Prove that one can touch them with spike such that booth spiked places maps same point on the landscape.

2. Find limit of the sequence

$$\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \dots$$

3. (Arnolds problem). Find the limit

$$\lim_{x\to 0} \frac{\sin(\operatorname{tg}(x)) - \operatorname{tg}(\sin(x))}{\arcsin(\operatorname{arctg}(x)) - \operatorname{arctg}(\arcsin(x))}$$

4 Iterations. Big n behavior.

1. A $\triangle ABC$ and point x_0 are given. On n-th step one choose one of the vertices of the $\triangle ABC$ join x_n with this vertice and replace point x_n to the point x_{n+1} which is midpoint of this line segment. Prove that there exist a figure S of arrear 0.0001 attracting all the points x_n for all sufficiently large n for all such processes.

2. $f(x) = x^2 - 10$. Prove that the set of all points x such that

$$\lim_{n\to\infty} f^{(n)}(x) \neq \infty$$

can be coved by finite set of intervals of total length 0.0001.

5 Iterations near stability points and circles.

1. Solve equations for each n in real numbers $f^{(n)}(x) = x$ if $f(x) = \cos(x)$.

2. Solve equations for each n in real numbers $f^{(n)}(x) = x$ if $f(x) = 1 - x^2$.

3.) Prove that $\sin^{(n)}(x) \to 0$ for $n \to \infty$.) Find $\lim_{n\to\infty} \sqrt{n} \sin^{(n)}(x_0)$.

4. Generalize previous problem for $f(x) = x - ax^k, k > 1$.

5. Generalize previous problem for $f(x) = x - \exp(-1/x^2)$.