7. Conjugacy and iterations

Functions f and g are called *conjugated* whenever $f = R \circ g \circ R^{-1}$ for some function R.

Exercise. Show that in this case $f^{(n)} = R \circ g^{(n)} \circ R^{-1}$.

We use notation $f \sim g$ whenever $\frac{f(x)}{g(x)} \xrightarrow{x \to 0} 1$.

1. a) Show that for any $n \in \mathbb{Z}$ the function $\cos(n \arccos(x))$ is a polynom, and that any two functions of this kind commute one with each other.
   b) Show that for any $n \in \mathbb{Z}$ the function $\sin((2n+1) \cdot \arcsin(x))$ is a polynom, and that any two functions of this kind commute one with each other.
   c) Show that for any $n \in \mathbb{Z}$ the function $\tan(n \arctan(x))$ is *rational*, i.e. is a quotient of two polynoms. Show that any two functions of this kind commute one with each other.

   Remark. Parts a and b provides nontrivial examples of commuting families of polynoms. By the deep theorem of Reed any other nontrivial family of commuting polynoms essentially coincides with a or b.

2. Show that the function $\sin x$ is not conjugated to a polynom.

3. Find fractional iterations of functions $\frac{ax+b}{cx+d}$ for any a, b, c, d.

   Hence we can explicitly describe fractional iterations of linear functions, we wish to connect them with as many fractional iterations of other functions as possible. Essentially we wish to find a big enough class of functions f for which exists a *conjugating function* R such that

   $$R \circ f \circ R^{(-1)}$$

   is a linear function. From time to time it is reasonable to find such conjugating function in some neighborhood of some point.

4. Let f be a function such that $f(0) = 0, f'(0) = k$. Evaluate $(f^{(n)})'(0)$. Assume that f is conjugated to lx for some number l with some smooth (i.e. infinitely differentiable) conjugating function R. Prove that in this case $k = l$. If $|k| < 1$ then $f^{(n)}(x) \xrightarrow{n \to \infty} 0$ in for all x from some neighborhood of 0. In this case we call 0 an *attracting* point of f. If $|k| > 1$ we call 0 a *repelling* point of f.

5. a) Let 0 be an attracting point of a continuously differentiable function f. Prove that for all x_0 from some neighborhood of 0 the limit

$$\lim_{n\to\infty} \frac{f^{(n)}(0)}{k^n} = G(x_0)$$

exists. Prove that G is continuous and that $G(k \cdot G^{(-1)}(x)) = f(x)$.
b) Prove that G is continuously differentiable.
c*) Prove that if f smooth then G is smooth.

6. Prove the following equality.

$$\frac{\sqrt{x}}{2} \cdot \frac{\sqrt{2+\sqrt{x}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{x}}}}{2} \cdots = \frac{4-x^2}{\sqrt{2\ln(\frac{x+\sqrt{x^2-4}}{2})}}.$$

Definition. Let $M \subset \mathbb{R}$ be a set. We call $x \in M$ a *limit point* of M if any neighborhood of x contains infinitely many points of M. We call the set $\{f^{(n)}(x)\}_{n\in\mathbb{Z}_{\geq 0}}$ an *orbit of x under the action of f*.

7. Let f, g be a pair of commuting smooth functions such that f(0) $=$ $g(0) = 0$ and $f(x) \sim x^\lambda, g(x) \sim x^\delta$. Then $f = g^{(\log_\delta \lambda)}$ (Here you have to define fractional iterations in a spirit of the introduction).

8. We call two points x_0, x_1 of an invertible function f *neighbor* if there exists a point x such that $\lim\limits_{n\to-\infty} f^{(n)}(x) = x_0$ and $\lim\limits_{n\to+\infty} f^{(n)}(x) = x_1$.

   Let x_0 and x_1 be common neighbor fixed points of commuting invertible continuously differentiable functions f and g. Assume x_0 is attracting and x_1 is repelling for both f and g. Prove that in this case

$$\log_{|g'(x_0)|} |f'(x_0)| = \log_{|g'(x_1)|} |f'(x_1)|.$$

9. Prove that functions of problems 6.1, 6.3, 6.5, 6.7 are fractional iterations of the corresponding functions.

10. Let f be a decreasing function such that $f(0) = 0$ and $f(x) \neq x$ for all $x \neq 0$. Does there exists an infinite family $\mathfrak{F}$ of pairwise noncommuting functions such that any element of $\mathfrak{F}$ commutes with f?

8. More about polynoms

1. Let $P(x)$ be a polynom of degree $n > 1$. Then for all m the set of polynoms of degree m such that they commute with f is finite.

2. Let $P(x)$ be a polynom of degree $n > 1$, $Q(x)$ be a polynom of degree $m > 1$ such that
   a) $P \circ Q = Q \circ P$, b) $P(x_0) = Q(x_0) = x_0$, c)$P'(x_0) > 1$, d) in any punctured neighborhood of x_0 there exists a point x_i such that $P^{(k)}(x_i) \xrightarrow{k \to \infty} \infty$.
   Prove that $P'(x_0)^{\log_n(m)} = Q'(x_0)$.

Remark. We note that the condition "to be commutative" is *algebraic*, i.e. is equivalent to a system of polynomial equations on coefficients. Therefore we could assume that the value of derivatives in all fixed point are algebraic. In assumption of transcendency of powers $\alpha^{\log_n(m)}$, where α is some algebraic number and n is not a rational power of α, we have that k is a rational power of n. It is interesting to derive the classification of commuting polynoms from this observation. This problem is really valuable because it provides a connection between dynamical systems, theory of transcendence numbers and theory of Diofant approximations.