Partitioning of a set into pieces of smaller diameter

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1 Definitions and notations

Let the *diameter* of a set Ω on the plane be the quantity

$$\operatorname{diam}\Omega = \sup_{\mathbf{x},\mathbf{y}\in\Omega} |\mathbf{x}-\mathbf{y}|.$$

Here

$$|\mathbf{x}-\mathbf{y}| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}, \quad \mathbf{x}=(x_1,x_2), \quad \mathbf{y}=(y_1,y_2),$$

i.e. this is ordinary “Euclidian” distance between points on the plane. The symbol “sup” means “supremum” (least upper bound), and it is convenient to suppose that “sup” means maximum. Essentially, diameter of a set is a maximal distance between its points. Why do we write not well known supremum nevertheless? Because there exist sets without a pair of points with maximal distance between them. For example, a disc without its boundary (remove its bounding circle). Never mind though. We consider only “closed” sets (which contain their boundary), so supremum is not necessary.

Consider an arbitrary bounded set Ω. We can suppose that $\operatorname{diam}\Omega = 1$ (zoom in or zoom out the set using homothetic transformation to obtain the desired diameter). We aim to sparingly dissect Ω into pieces of smaller diameter. In other words, we aim to represent Ω in the form

$$\Omega = \Omega_1 \cup \ldots \cup \Omega_f$$

under the condition $\operatorname{diam}\Omega_i < \operatorname{diam}\Omega$ for all i and we aim to minimize f. You can imagine Ω as some crooked cake which does not go through our mouth as a whole but which we desire to eat greedily, to bite the least number of times. And so we should dissect a cake into pieces such that number of these pieces is the least and each piece fits in the mouth still.

The main question: which cake is the worse in the sense of the above-described problem about dissecting? In other words, into how much pieces can we dissect any cake deliberately?

K.Borsuk in 1933 formalized the problem described above: what is the least number $f(2)$ such that any bounded set Ω on the plane admits a partition into $f(2)$ parts of smaller diameter? Of course, Borsuk was not reflecting about crooked cakes and even topology served as the motivation for his research. More detailed and, by the way, more intriguing history of the problem can be found in [1].

One more question: why do we write $f(2)$? The answer is that the plane is two-dimensional. In the truth, Borsuk raised a similar problem on the real axis too (corresponding quantity is $f(1)$), and in space of any dimension also. Usually the space of dimension n is denoted by $\mathbb{R}^n$. In particular, $\mathbb{R}^1 = \mathbb{R}$ is the real axis, $\mathbb{R}^2$ is the plane, $\mathbb{R}^3$ is the space in common sense, i.e. 3-dimensional space in which we live. In general case $f(n)$ is a *Borsuk number*, which is the least f, such that any bounded set in $\mathbb{R}^n$ can be dissected into f pieces of smaller diameter, but there exists a bounded set in $\mathbb{R}^n$, which cannot be disssected into $f-1$ pieces of smaller diameter. Here the distance between points in $\mathbb{R}^n$ is standard (is measured “by Euclid”):

$$|\mathbf{x}-\mathbf{y}| = \sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}, \quad \mathbf{x}=(x_1,\ldots,x_n), \quad \mathbf{y}=(y_1,\ldots,y_n).$$

Borsuk supposed that $f(n) = n+1$. This supposition is called *the Borsuk conjecture*. This conjecture was disproved in 1993 (see [1], [2]), but it remains a great amount of questions without answers still.

In the following section we introduce the problems. Step by step you approach the interesting problems in which an advance can be done by elementary methods.

2 Problems before intermediate finish

Problem 1. Prove that $f(1) = 2$.

Problem 2. Prove that $f(2) \geqslant 3$. In other words, give an example of a set on the plane which cannot be dissected into two pieces of smaller diameter.

Problem 3. Dissect a square into pieces of smaller diameter.

Problem 4. Dissect a disc of radius $1/2$ into three pieces of smaller diameter.

Problem 5. Dissect a disc of radius $1/2$ into three pieces of diameter not greater than $\frac{\sqrt{3}}{2} = 0.866...$

Problem 6. Prove that the constant $\frac{\sqrt{3}}{2}$ from the problem 5 is unimprovable, i.e. for any partition of a disc of radius $1/2$ into 3 pieces at least one of those pieces has diameter not smaller than $\frac{\sqrt{3}}{2}$.

Call by *universal cover* in $\mathbb{R}^n$ such set Ω that any set Φ of unit diameter can be moved into Ω, i.e. it is possible to cover Φ by Ω. For instance:

Problem 7. Prove that a unit square is a universal cover on the plane.

Problem 8. Prove that a unit cube is a universal cover in the space of any dimension. Here we call by unit cube a set of points $\mathbf{x} = (x_1, \ldots, x_n)$, such that $x_i \in [0, 1]$ for all i.

It is known that a regular hexagon Ω_6 with distance 1 between its parallel sides is a universal cover on the plane. For detailed proof see [3]. However, you may reflect on it.

Problem 9. Dissect the regular hexagon Ω_6 into three parts of diameter $\frac{\sqrt{3}}{2}$. Deduce from this problem the validity of the Borsuk conjecture on the plane and, in some sense, "unimprovability" of the constant $\frac{\sqrt{3}}{2}$ from the problem 6.

Problem 10. What number plays the role of unimprovable constant from the problem 9 on the real axis?

Problem 11. Prove that in any set of n points on the plane there are at most n pairs of points such that the distance between them equals the diameter of the set.

Problem 12. Deduce form the problem 11 the validity of the Borsuk conjecture for *finite* sets on the plane (not involving universal covers).

Problem 13*. Prove that a disc of radius $\frac{1}{\sqrt{3}}$ is a universal cover on the plane.

Problem 14. Explain why a disc of radius $r < \frac{1}{\sqrt{3}}$ cannot be a universal cover.

Problem 15. Explain why the result of the problem 13* does not allow to prove the Borsuk conjecture on the plane.

Problem 16. A disc B_1 of raduis $\frac{1}{\sqrt{3}}$ is given on the plane. Choose an arbitrary point on its boundary and consider the disc B_2 of radius 1 with center in chosen point. Prove that $B_1 \cap B_2$ is a universal cover on the plane.

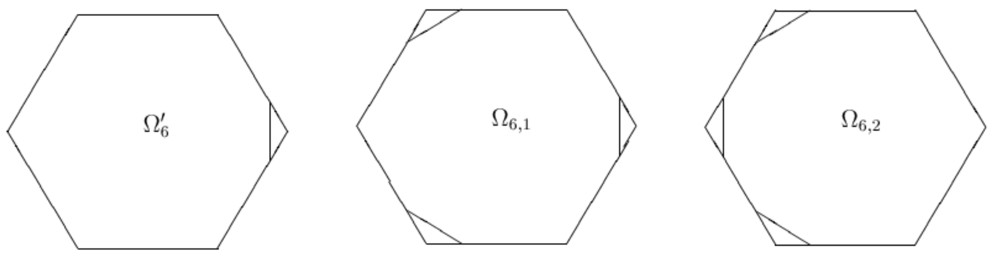
Problem 17. Using the result of the problem 16 prove the Borsuk conjecture.

Problem 18* (research). Is it possible to prove using the result of the problem 16 that any set of diameter 1 can be dissected into three pieces of diameter $\leqslant \frac{\sqrt{3}}{2}$? What is the least such constant that you obtained?

Call by *univesal covering system (ucs)* in $\mathbb{R}^n$ such collection of sets $\{S_\alpha\}$, that any $\Omega \subset \mathbb{R}^n$, $\operatorname{diam} \Omega = 1$, can be moved into at least one of the sets S_α.

Problem 19. Consider the regular hexagon Ω_6 with distance 1 between parallel sides. Consider a segment connecting the center of hexagon with one of its vertices and draw a line perpendicular to this segment at the distance $1/2$ from the center. This line cuts off a triangle from the hexagon. Prove that the hexagon without mentioned triangle also is a universal cover on the plane. This truncated hexagon is denoted by Ω_6' on the picture 1.

Problem 20. Prove that the middle and the regular hexagons form ucs. Triangles, as in the problem 19, are cut off by the lines at the distance $1/2$ from the hexagon center and perpendicular to the segments connecting the center with corresponding vertices.



Picture 1: Examples of ucs.

Problem 21. Prove that any set of diameter 1 on the plane can be dissected into 5 pieces without a pair of points at the distance $\frac{1}{\sqrt{3}}$. **Hint.** Use Ω_6.

Problem 22* **(research).** Is it possible to prove that for some $a < \frac{1}{\sqrt{3}}$ any set of diameter 1 on the plane can be dissected into 5 pieces without a pair of points at the distance a?

Problem 23. Find such n, that any set on the plane can be dissected into n pieces without a pair of points at the distance 1. What is the least n that you found?

Problem 24*. Prove that any set of diameter 1 on the plane can be dissected into 6 pieces of diameter not greater than $\sqrt{\frac{13}{3}(2-\sqrt{3})} = 0.5577...$ **Hint.** Use ucs $\{\Omega_{6,1}, \Omega_{6,2}\}$.

Problem 25.** Even better than in the problem 24*?

Problem 26*. Prove that any set of diameter 1 on the plane can be dissected into 5 pieces of diameter not greater than 0.603. **Hint.** Use the universal cover Ω_6'.

Problem 27.** And even beter than in the problem 26*?

Problem 28. Prove that any set of diameter 1 on the plane can be dissected into 4 pieces of diameter not greater than $\frac{1}{\sqrt{2}}$.

Problem 29. Prove the unimprovability of the result of the problem 28.

Problem 30. Prove that any set of diameter 1 on the plane can be dissected into 7 pieces of diameter not greater than $\frac{1}{2}$.

Problem 31. Prove the unimprovability of the result of the problem 30.

3 Problems after intermediate finish

Problem 32. Prove that $f(3) \geqslant 4$.

Problem 33*. Prove that a ball with radius $\sqrt{\frac{3}{8}}$ is a universal cover in $\mathbb{R}^3$.

Problem 34. Explain why a ball of radius $r < \sqrt{\frac{3}{8}}$ cannot be a universal cover in $\mathbb{R}^3$.

Problem 35. Split a ball of radius $1/2$ into four pieces of smaller diameter in the space (i.e. diameter of each piece must be smaller than 1).

Problem 36*. Let regular tetrahedron be inscribed in the ball of radius $1/2$. Let us consider 4 trihedral angles which we can get if we connect the center of the ball with vertices of tetrahedron's faces. Let us intersect each of these angles with the ball. So we will have a partitioning of our ball into 4 equal pieces. Find diameters of these pieces.

Problem 37. Let us define *regular simplex* in $\mathbb{R}^n$ as an analog of regular triangle on the plane and regular tetrahedron in the space. Notably let us consider $n+1$ points $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ in $\mathbb{R}^n$ such that $|\mathbf{x}_i - \mathbf{x}_j| = a$ for every pairs $i \neq j$ and some $a > 0$. Prove the existence of the simplex.

Problem 38*. Let us define *n-dimensional ball* as a set

$$B = \{\mathbf{x} = (x_1, \ldots, x_n) : \ x_1^2 + \ldots + x_n^2 \leqslant 1\}.$$

It is a ball with radius 1 and with diameter 2. Let us inscribe the regular simplex in this ball. Find the length of the side of this simplex.

Problem 39.** Let us realize partitioning of the ball from the problem 38* like partitioning from the problem 36*. Notably let us inscribe a regular simplex whose sides' length was found in problem 38* in our ball and let us consider such polyhedral angles which have a vertex in the center of our ball and pass through the simplex faces (there are $n+1$ faces of simplex). Find diameters of obtained pieces. In particular ascertain that they are smaller than 2 and also that their values tend to 2 with $n \to \infty$.

Problem 40. Let us intersect the ball B_1 of radius $\sqrt{\frac{3}{8}}$ with any ball B_2 of radius 1 whose center lies on the border of the ball B_1. Prove that $B_1 \cap B_2$ is a universal cover in $\mathbb{R}^3$.

Problem 41*. Prove that $B_1 \cap B_2$ can be split into 5 pieces of diameter smaller than 1.

Problem 42. Prove that $f(4) \geqslant 5$. Also in the general case $f(n) \geqslant n+1$.

Problem 43. Find any upper bound for $f(4)$.

Problem 44. Find any upper bound for $f(n)$.

Problem 45. Construct a cover in $\mathbb{R}^4$, like covers from the problem 40 of this list and also the problem 16 from the list of problems before the intermediate finish.

Problem 46*. Prove that the cover from the problem 45 can be split into 9 pieces of diameter smaller than 1 and so $f(4) \leqslant 9$.

4 Solutions

When solving the problems of the project, some of the school participants have obtained brilliant and unexpected results. First of all, it is worth making special mention of a result by Egor Voronetskiy who succeeded in improving all the previously known upper bounds for the minimum value of a forbidden distance between any two monochromatic points in a colouring of an arbitrary planar set of diameter 1 with four and five colours. If the previous bounds were by the values $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}}$, then now, due to Egor, we have bounds by the values $\frac{1}{\sqrt{3}}$ and $\frac{1-\sqrt{3}+\sqrt{6\sqrt{3}}}{4\sqrt{3}-2}$. Egor's constructions are given below. Also we want to say about a new upper bound for the minimum diameter in a partitioning of an arbitrary planar set of diameter 1 into six parts. This bound was independently obtained by Dima Belov with Nikita Aleksandrov and by How Si Wei. If the previous bound had a value $0.557\ldots$, then the current bound equals $0.542\ldots$ Below, we also present these constructions. At the same time, some school participants have shown a very high mathematical culture: they managed to carefully calculate the diameters of parts in a dissection of the n-dimensional unit ball into $n+1$ parts (problem 8^{**} from the second part of the project); they also managed to give a proof of a theorem by M. Lassak, which was published in 1982 and which asserts that any set of diameter 1 in $\mathbb{R}^n$ can be partitioned into $2^{n-1}+1$ pieces of diameter < 1. Both the dissection of the unit ball and the proof of Lassak's theorem were done by Egor Voronetskiy and Maksim Didin.

1. It is obvious that $f(1) > 1$. It is sufficient to dissect the segment into 2 pieces with smaller diameter as long as for any set with diameter 1 on the line we can put it in the segment with length 1. For example we can dissect our segment in halves.

2. It is equilateral triangle for example.

3. We can do it with one vertical section for example.

5. We can cut our circle by three radii, with angles of $120°$ between them.

6. Let our circle be dissected into three pieces. Let us take a point on the boundary of our circle so that it lies in two of these pieces. And let us construct an equilateral triangle with one vertex at our point and other vertices on the boundary of our circle. Then two vertices of this triangle are in the same piece. So the diameter of this piece is not less than $\frac{\sqrt{3}}{2}$.

8. It is obvious that the set Ω with diameter 1 is in stripe $\min\limits_{x\in\Omega} x_i \leqslant x_i \leqslant \max\limits_{x\in\Omega} x_i$ with width at most 1. Intersection of such stripes for all i is a parallelepiped. We can cover it by the unit cube.

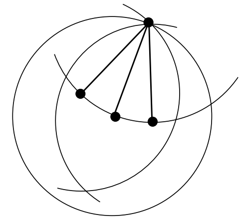
9. Dissect the regular hexagon into three pieces by perpendiculars from the center to not adjacent sides. The diameter of each piece equals $\frac{\sqrt{3}}{2}$. Then place the set Ω of diameter 1 into Ω_6. The partition of Ω_6 provides the partition of Ω into the pieces of diameter at most $\frac{\sqrt{3}}{2} < 1$. And as we know from the problem 6, the constant $\frac{\sqrt{3}}{2}$ is unimprovable in the case of the circle of diameter 1.

10. Any set of diameter 1 on the line can be dissected into two pieces of diameter $\frac{1}{2}$ but the unit segment cannot be dissected into two parts of diameter smaller than $\frac{1}{2}$.

11. Connect by edges all pairs of points at the distance 1. If every point is incident to at most two edges then we have at most n edges. So consider a triple of edges with common end. Since the distance between points is at most 1, the angle between edges is at most $60°$.

Another end of the middle edge is incident only to this edge, since any two points are at the distance at most 1 (see pic.2). So erase the middle edge with its not common end and continue the proof by induction.

12. It is sufficient to paint in three colours the vertices of the graph from the previous problem. Do it by induction. Suppose you have a vertex v with only one edge incident to it. Erase v, colour the rest graph by induction and then paint v not in the colour of its neighbour. In other cases the graph is a collection of cycles. Paint them in three colours.



Picture 2: Three edges with common end.

13. *The first solution.* Place the set Ω of diameter 1 into a large disc B. Then reduce this disc in such a way that it always contains Ω and once some points appear on the boundary of B they continue to lie there. We cannot reduce the disc if one of two cases occurs:

1) ends of some diameter of B belong to Ω, so the diameter of B is not greater than 1;

2) some three points of Ω lie on the boundary of B and form an acute-angled triangle, it is well-known that in this case the diameter of B is not greater than $\frac{1}{\sqrt{3}}$.

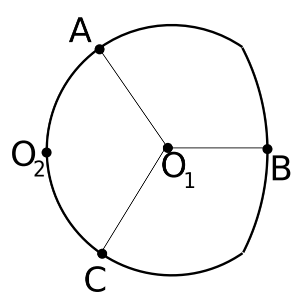
The second solution. Since the regular hexagon inscribed in the circle is a universal cover, this is a universal cover also.

14. Suppose that the equilateral triangle with the side 1 is contained in a disc with radius r, then reduce this disc as in the solution of the previuos problem. In the end we obtain the circumscribed disc about the triangle. So $r \geqslant \sqrt{\frac{1}{3}}$.

15. By the problem 6 if the disc with radius $\frac{1}{\sqrt{3}}$ is dissected into three parts then the diameter of some part is at least 1.

16. By the problem 13 we can place any set Ω of diameter 1 into the circle B_1 with radius $\frac{1}{\sqrt{3}}$. Then translate this set so that the point $X \in \Omega$ appears on the boundary of the circle but Ω continues to lie in B_1. Draw a circle B_2 with radius 1 and center X. Since the diameter of Ω equals 1, B_2 covers Ω also. So the intersection $B_1 \cap B_2$ covers Ω.

17. The universal cover from the problem 16 can be dissected into three parts of diameter $c = \frac{\sqrt{3}}{2} + \frac{\sqrt{\frac{2}{\sqrt{3}}-1}}{2} < 1$. We see on the picture that O_1 is the center of circle with radius $\frac{1}{\sqrt{3}}$, O_2 is the center of the circle with radius 1, B is the middle of the arc with unit radius and points A and C are chosen in such a way that the triangle ABC is equilateral. It is easy to compute that its side equals $\frac{\sqrt{3}}{2} + \frac{\sqrt{\frac{2}{\sqrt{3}}-1}}{2}$.



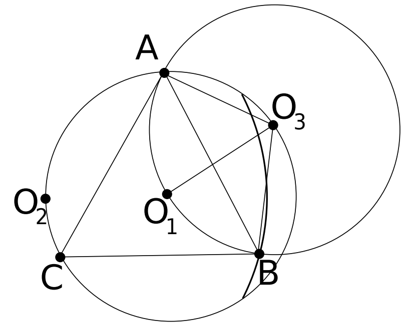
Picture 3: The partition of the universal cover from the problem 16.

After that the Borsuk conjecture can be proved in a similar way as in the problem 9.

18. Let us prove that $B_1 \cap B_2$ cannot be dissected into three parts of diameter smaller than c which is defined in the solution of the previous problem.

Lemma. For every point X on the boundary of $B_1 \cap B_2$ there exists an equilateral triangle whose vertices belong to the boundary of $B_1 \cap B_2$ and one of them is X. Also the side of such triangle is not shorter than c.

Proof. Rotate the boundary of $B_1 \cap B_2$ about X by 60 degree clockwise and counter-clockwise. Intersect the image under rotation with boundary of $B_1 \cap B_2$. Intersection points are the vertices of the equilateral triangle. Exatly one of them lies on the arc with radius 1: if no points lie on this arc then we have an inscribed angle $< 60°$, and if two points lie on this arc — $< 60°$. On the picture there are O_1 — the center of the circle with radius $\frac{1}{\sqrt{3}}$, O_2 — the center of the circle with radius 1, ABC — the equilateral triangle, where vertex B lies on the arc with radius 1.



Picture 4: A equilateral triangle on the boundary of the universal cover from the problem 16.

Rotate the circle with center O_1 about A by $60°$. Denote by O_3 its center. Clearly points O_1 and B lie on the circle with center O_3, i.e. $O_3A = O_3B = \frac{1}{\sqrt{3}}$. Therefore, AB decreases if the angle AO_3B decreases. But $BO_3O_1 = 60°$. So the angle AO_3B decreases as the angle O_1O_3B decreases. Also $O_3O_1 = O_3B = \frac{1}{\sqrt{3}}$. So the angle O_1O_3B decreases as O_1B decreases. O_1B is minimal if B is the middle of the arc with radius 1. Therefore, $AB \geqslant c$.

Return to the proof of the initial problem. Suppose $B_1 \cap B_2$ is dissected into three parts. Consider the point X on the boundary which belongs to 2 parts. Apply lemma to point X. In the triangle from lemma some two vertices belong to the same part, i.e. the distance between them is at least c. Therefore the diameter of this part is at least c.

19. Consider two small triangles that lie in the opposite vertices of the hexagon. The distance between any two points of these triangles is at least 1. So, if a set of diameter 1 lies within the hexagon then one of these triangles can be cut off without touching the set.

20. Continuing the proof of the previuos problem we can cut off a triangle in every pair of opposite triangles without touching the set. So we obtain one of the hexagons on the picture.

21. *Solution is given by the participant of the conference Egor Voronetskii.* It is sufficient to dissect the regular hexagon with the length of the side $\frac{1}{\sqrt{3}}$. However it can be dissected into 4 parts!

In the center of the picture the Reuleaux triangle of diameter $\frac{1}{\sqrt{3}}$ is placed. It is easy to compute that $AB = \frac{1}{\sqrt{3}}$. Colour the arc CD in green, but the vertex C — in blue. Colour arcs AC and BC in blue, but vertices A and B — in red and yellow respectively. Colours of other arcs and vertices are defined using the rotation by $120°$.

22. *Solution is given by the participant of the conference Egor Voronetskii.*

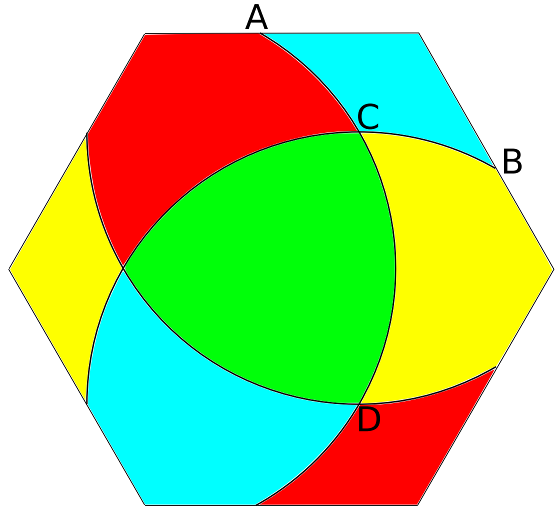
Dissect the hexagon similarly to the previous problem, but the diameter of the Reuleaux triangle equals AB in this case (see pic.6).

We change color only in the interior of the blue part. Other points keep their color.

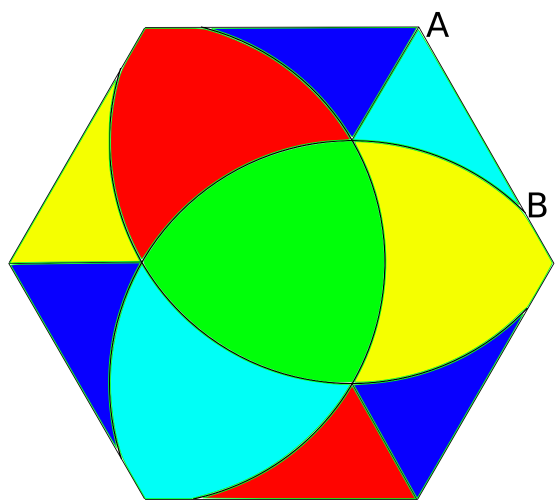
23. This problem is equivalent to the problem about the chromatic number of the plane which is unsolved. It is known only how to paint the plane in 7 colours: hexagons of regular hexagonal lattice can be painted in 7 colours in such a way that adjacent hexagons are of different colour.

24. Let $\sigma = \sqrt{\frac{13}{3}\left(2 - \sqrt{3}\right)}$. We will dissect hexagons $\Omega_{6,1}$ and $\Omega_{6,2}$ into 6 pieces of diameter not greater than σ. By the problem 20 it is sufficient for the proof.

Let us start with hexagon $\Omega_{6,1}$.



Picture 5: There are no points of the same colour at the distance $\frac{1}{\sqrt{3}}$



Picture 6: There are no points of the same colour at the distance $\frac{1-\sqrt{3}+\sqrt{6\sqrt{3}}}{4\sqrt{3}-2}$.

Suppose that $\Omega_{6,1}$ is obtained from the regular hexagon $ABCDEF$ by cutting triangles BB_1B_2, DD_1D_2 and FF_1F_2. Let $M_1, \ldots, M_6$ be the middle points of the sides and O be the center (see pic. 7). Consider segments AX, CY, EZ of length $1/2$ on rays AO, CO, EO and draw segments OX, OY, OZ. Then connect all points X, Y, Z with two sides of the hexagon (see pic.7). So we obtain a partition of $\Omega_{6,1}$ into six polygons $OXM_1B_2B_1M_2Y$, AM_1XM_6 etc.

Their diameters are not greater than σ.

Indeed, $AX = M_1M_6 = \frac{1}{2}$. Therefore the diameter of quadrilateral AM_1XM_6 equals $\frac{1}{2}$. The diameter of 7-gon $OXM_1B_1B_2M_2Y$ equals $OB_1 = \frac{1}{2\cos\frac{\pi}{12}} \approx 0.5176$. Other diameters can be obtained from these two ones by rotation of the picture.

Now dissect the hexagon $\Omega_{6,2}$ in the following way.

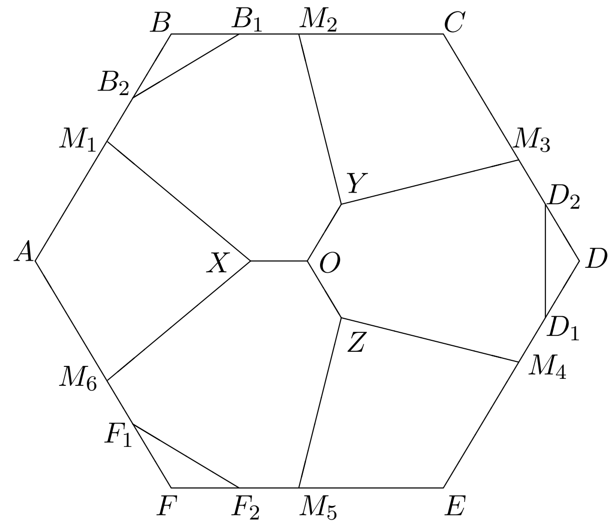
Suppose that $\Omega_{6,2}$ is obtained from the hexagon $ABCDEF$ by cutting triangles AA_1A_2, BB_1B_2 and FF_1F_2 (see pic. 8). Let M be the middle of A_1A_2, and J be the intersection of a midperpendicular to the segment A_1C with segment AD. Clearly $JA_1 = JC$.

Draw lines parallel to diagonals BE and CF through the point J. Suppose these lines intersect $\Omega_{6,2}$ by segments PQ and RS (see pic. 8). So we have $\Omega_{6,2}$ dissected into six polygons.

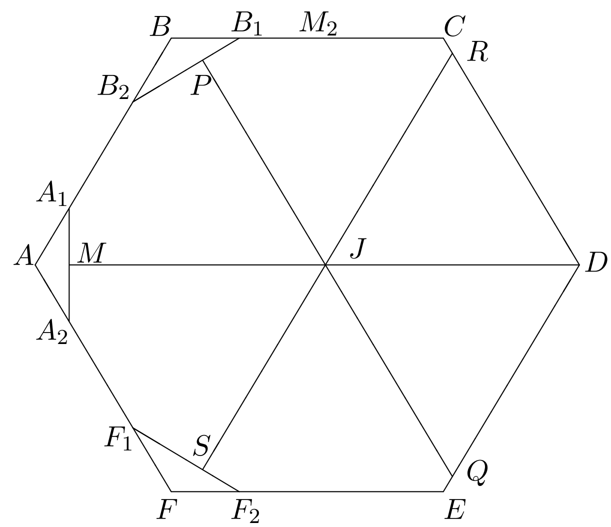
Let us check that their diameter is not greater than σ.

Note that $JA_1 = JC = \sigma$.

Indeed, by Pythagorean theorem for triangles JMA_1 and JM_2C (the point M_2 is the middle of BC) the



Picture 7:



Picture 8:

equality $JA_1^2 = JC^2$ is equivalent to

$$\left(OJ + \frac{1}{2}\right)^2 + \left(\frac{\operatorname{tg}\frac{\pi}{12}}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{3}} - OJ\right)^2,$$

so we obtain

$$OJ = \frac{1 - 3\operatorname{tg}^2\frac{\pi}{12}}{4(3+\sqrt{3})}.$$

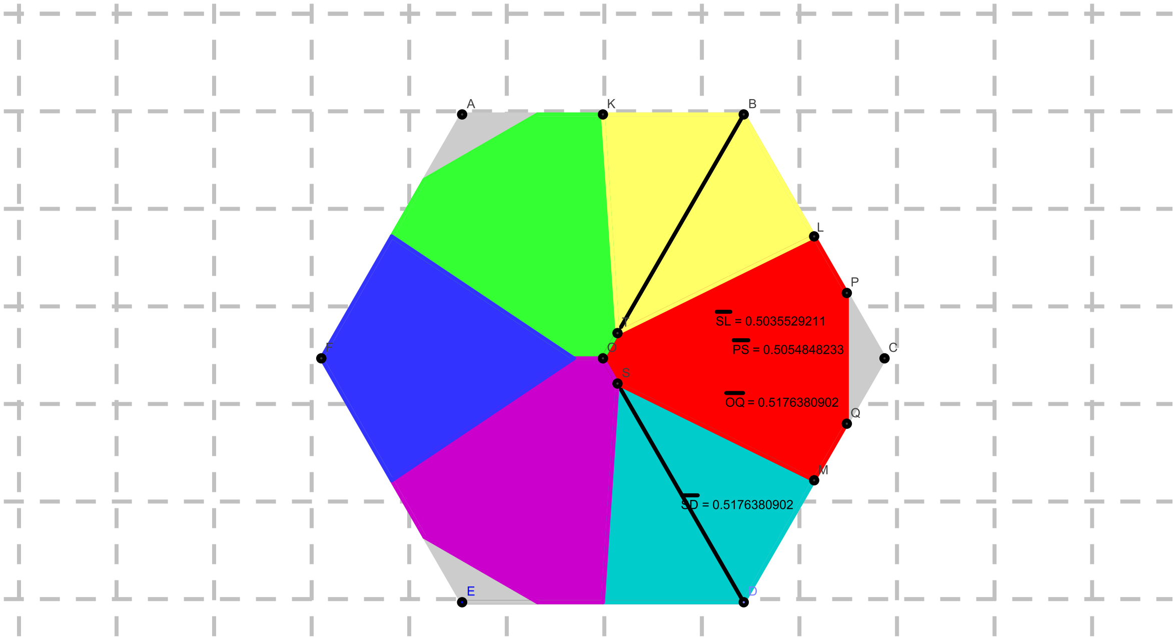
Now the length of the segment JA_1 can be computed by Pythagorean theorem for the triangle JMA_1.

It is not so difficult to compute that diameters of pentagons JMA_1B_2P and JPB_1CR equal $JA_2 = JC = \sigma$, and diameter of the equilateral triangle JRD is less than σ. The partition is symmetric with respect to the line MD. This finishes the proof.

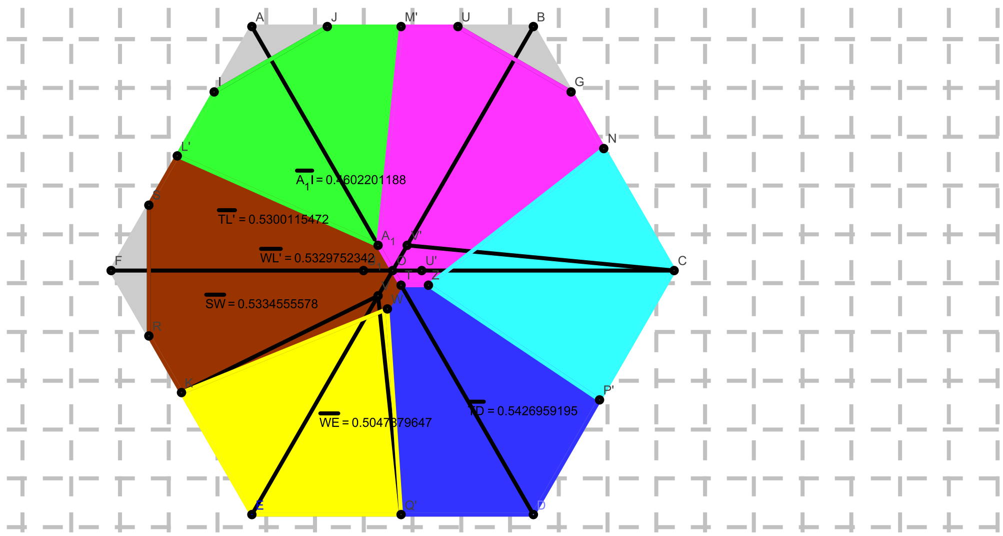
25. *Solution is given independently by two teams: Aleksandrov Nikita and Belov Dima, and How Si Wei.*

Dissect elements of ucs. At first dissect the polygon on the picture 9. It is symmetric with respect to rotation by $120°$ and $BT = \sqrt{2-\sqrt{3}}$. We have diameters of all parts equal $\sqrt{2-\sqrt{3}}$.

Then consider the picture 10. The point T lie on the diagonal AD and $ST = TD$. Also $AA_1 = EV = CU' = \sqrt{2-\sqrt{3}}$. The point W is chosen in such a way that W lies on VD and TW is parallel to CD. The point Z can be defined in similar way. Points M', N, P', Q', K and L' are bases of perpendiculars dropped from the point T. After some computations we see that diameters of all parts are at most $TD = \frac{10-4\sqrt{3}}{5\sqrt{3}-3} \approx 0,5427$.



Picture 9: Diameters of all parts equal $\sqrt{2-\sqrt{3}}$.



Picture 10: Diameters of all parts are at most TD.

26. Dissect Ω_6' into five sets in the following way.

Consider points X, Y, Z and T on the sides BC, CD, DE EF respectively such that all sides of the pentagon $MXYZT$ are equal (see pic. 11).

Clearly this can be done in the unique way.

Denote the length of the segment MX by ρ_5'. It is easy to compute that $\rho_5' = 0.6020\ldots$.

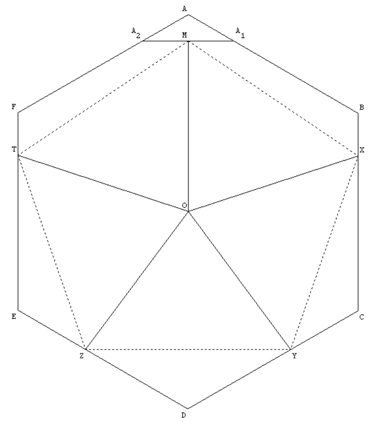
Connect the point O with all vertices of $MXYZT$. So Ω_6' is dissected.

Obviously the distance between the point O and arbitrary point on the boundary of Ω_6' is not greater than $\frac{1}{\sqrt{3}} < \rho_5'$. Therefore parts of the partition have diameter ρ_5'.

27. ?

28. By the problem 7 the partition of the unit square into 4 parts of diameter $\frac{1}{\sqrt{2}}$ is sufficient. Draw the diagonals.

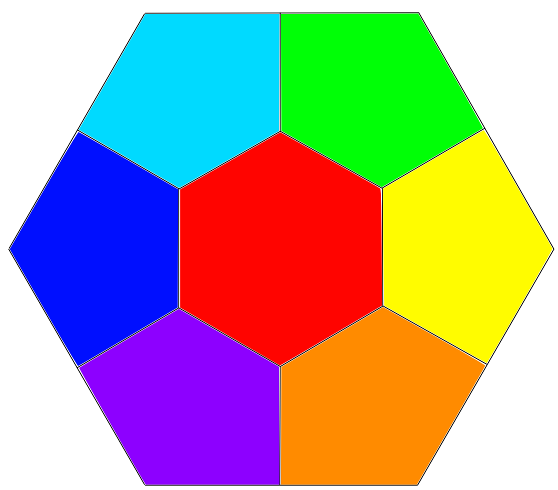
29. Let us prove that the circle of diameter 1 cannot be dissected into four pieces. Suppose the contrary. Let us take a point on the boundary of our circle so that it lies in two of these pieces. And let us construct a square



Picture 11:

with one vertex at our point and other vertices on the boundary of our circle. Then two vertices of this square are in the same piece. So the diameter of this piece is not less than $\sqrt{\frac{1}{2}}$.

30. Since the regular hexagon with side $\frac{1}{\sqrt{3}}$ is a univeral cover, its partition into 7 parts of diameter $\frac{1}{2}$ is sufficient (see pic.12)



Picture 12: the maximal diagonals of the hexagon and pentagons equal $\frac{1}{2}$.

31. Let us prove that the circle of diameter 1 cannot be dissected into seven pieces. Suppose the contrary. Let us take a point on the boundary of our circle so that it lies in two of these pieces. And let us construct a regular hexagon with one vertex at our point and other vertices on the boundary of our circle. Then two vertices of this hexagon are in the same piece. So the diameter of this piece is not less than $\frac{1}{2}$.

32. The regular tetrahedron cannot be dissected into three pieces of smaller diameter, because its vertices must lie in different pieces.

33. The solution is similar to the first solution of the problem 13. Place the set Ω of diameter 1 into a large ball B. Then reduce this ball in such a way that it always contains Ω and once some points appear on the boundary of B they continue to lie there. We cannot reduce the ball if one of three cases occurs:

1) ends of some diameter of B belong to Ω, so the diameter of B is not greater than 1;

2) some three points of Ω lie on the equator of B and form an acute-angled triangle, in this case the diameter of B is not greater than $\frac{1}{\sqrt{3}}$;

3) some four points of Ω lie on the boundary of B and form tetrahedron which contains the center of B.

It is sufficient to prove that in the third case the radius of the ball is not greater than $\sqrt{\frac{3}{8}}$. This follows from the lemma.

Lemma. Suppose a tetrahedron T is inscribed into a unit sphere and contains the center of the sphere. Then some side of T is not shorter than $\sqrt{\frac{8}{3}}$. Suppose additionaly that T is regular. Then its sides equal $\sqrt{\frac{8}{3}}$.

Proof. Let O be the center of the sphere and A, B, C, D be vertices of T. T contains O, therefore

$$a \cdot \overrightarrow{OA} + b \cdot \overrightarrow{OB} + c \cdot \overrightarrow{OC} + d \cdot \overrightarrow{OD} = \overrightarrow{0}$$

for some positive numbers a, b, c, d. Let a be the maximal number among them. Consider the inner product of vector $\overrightarrow{OA}$ with both parts of this equality:

$$a \cdot \overrightarrow{OA} \cdot \overrightarrow{OA} + b \cdot \overrightarrow{OB} \cdot \overrightarrow{OA} + c \cdot \overrightarrow{OC} \cdot \overrightarrow{OA} + d \cdot \overrightarrow{OD} \cdot \overrightarrow{OA} = 0.$$

Number a is maximal, so $\overrightarrow{OA} \cdot \overrightarrow{OX} < -\frac{1}{3}\overrightarrow{OA} \cdot \overrightarrow{OA}$ where $X = B, C$ or D. Without loss of generality $X = B$. By cosine law $AB \geqslant \sqrt{\frac{8}{3}}$ since $OA = OB = 1$. The inequality is exact in the case of the regular tetrahedron.

34. Suppose that the regular tetrahedron with the side 1 is contained in a ball with radius r, then reduce this ball as in the solution of the previuos problem. In the end we obtain the circumscribed ball about the tetrahedron. So $r \geqslant \sqrt{\frac{3}{8}}$.

36. Let A_1, A_2, A_3, A_4 be the vertices of the tetrahedron and O be its center. Consider the piece associated to the trihedral angle $OA_1A_2A_3$. Let B be the middle of A_2A_3. Draw the radius OB. Denote by C its end lying on the sphere. The statement is that the diameter equals the length of A_1C. We leave the proof of this fact to the reader.

By the previous problem a side of the tetrahedron equals $\sqrt{\frac{2}{3}}$. By the Pythagorean theorem for triangles A_1BA_2 and OBA_2 obtain that $|A_1B| = \sqrt{\frac{1}{2}}$ and $|OB| = \sqrt{\frac{1}{12}}$. By the cosine law a cosine of the angle A_1OB equals $-\sqrt{\frac{1}{3}}$.

By the cosine law for triangle A_1OC, obtain that

$$|A_1C| = \sqrt{\frac{3+\sqrt{3}}{6}} \approx 0.888... < 1.$$

37. n points which have one unit coordinate and whose all other coordinates are zero form a regular simplex in the plane $x_1 + x_2 + \dots + x_n = 1$.

39. Recall some geometric notions. *The inner product* of vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ is a number $(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2$. The distance $|\mathbf{x} - \mathbf{y}|$ between points $\mathbf{x}, \mathbf{y}$ is measured by formula

$$|\mathbf{x} - \mathbf{y}|^2 = (\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) - 2(\mathbf{x}, \mathbf{y}). \tag{1}$$

Notation $(\mathbf{x}, \mathbf{x})$ is called *the inner square* of the vector $\mathbf{x}$. It represents a square of the length $|\mathbf{x}|$ of this vector. The cosine of the angle between the vectors $\mathbf{x}, \mathbf{y}$ can be computed by formula $\frac{(\mathbf{x},\mathbf{y})}{|\mathbf{x}|\cdot|\mathbf{y}|}$. So the equality (1) is just the cosine law:

$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}| \cdot |\mathbf{y}| \cdot \cos(\widehat{\mathbf{x}, \mathbf{y}}). \tag{1$'$}$$

Say the same about $\mathbb{R}^d$. The inner product of vectors $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d)$ equals

$$(\mathbf{x}, \mathbf{y}) = x_1y_1 + \ldots + x_dy_d.$$

Let us compute the diameter of the part D associated to the vertices $\mathbf{x}_1, \dots, \mathbf{x}_d$ of the simplex T. Clearly $\mathbf{x}_1 + \ldots + \mathbf{x}_{d+1} = \mathbf{0}$ (see two- and three-dimensinal cases). Therefore

$$(\mathbf{x}_1, \mathbf{x}_i) + \ldots + (\mathbf{x}_{d+1}, \mathbf{x}_i) = (\mathbf{0}, \mathbf{x}_i) = 0$$

for each i. Since $|\mathbf{x}_1| = \ldots = |\mathbf{x}_{d+1}| = 1$ (all vertices of T lie on the unit sphere), we obtain that $(\mathbf{x}_i, \mathbf{x}_i) = 1$. Finally by symmetry all angles $(\widehat{\mathbf{x}_j, \mathbf{x}_i})$ for $i \neq j$ are the same. Hence for $i \neq j$ we obtain that

$$0 = (\mathbf{x}_1, \mathbf{x}_i) + \ldots + (\mathbf{x}_{d+1}, \mathbf{x}_i) = 1 + d(\mathbf{x}_j, \mathbf{x}_i),$$

i.e. $(\mathbf{x}_j, \mathbf{x}_i) = -\frac{1}{d}$. for all $i \neq j$

By the last observation and the cosine law (the equality (1′))

$$|\mathbf{x}_i - \mathbf{x}_j|^2 = 1 + 1 - 2 \cdot 1 \cdot 1 \cdot \left(-\frac{1}{d}\right) = 2 + \frac{2}{d}, \quad |\mathbf{x}_i - \mathbf{x}_j| = \sqrt{\frac{2d+2}{d}}.$$

In particular, substituting $d = 2$ or $d = 3$ we get $\sqrt{3}$ or $2\sqrt{\frac{2}{3}}$

Return to the diameter computing. We have two cases: $d = 2k$ and $d = 2k - 1$.

Case 1. Consider the points $\xi = \mathbf{x}_1 + \ldots + \mathbf{x}_k$ and $\eta = \mathbf{x}_{k+1} + \ldots + \mathbf{x}_{2k}$. These points do not belong to D if $k > 1$ but have some interest. So

$$\cos(\xi \hat{,} \eta) = \frac{(\mathbf{x}_1 + \ldots + \mathbf{x}_k, \mathbf{x}_{k+1} + \ldots + \mathbf{x}_{2k})}{|\mathbf{x}_1 + \ldots + \mathbf{x}_k| \cdot |\mathbf{x}_{k+1} + \ldots + \mathbf{x}_{2k}|} = \frac{(\mathbf{x}_1 + \ldots + \mathbf{x}_k, \mathbf{x}_{k+1} + \ldots + \mathbf{x}_{2k})}{|\mathbf{x}_1 + \ldots + \mathbf{x}_k|^2}.$$

The numerator in the last expression (after expansion) is a sum of k^2 summands of the form $(\mathbf{x}_i, \mathbf{x}_j)$ for all i and j. Therefore, numerator equals $k^2 \cdot \left(-\frac{1}{2k}\right) = -\frac{k}{2}$. Rewrite the denominator:

$$|\mathbf{x}_1 + \ldots + \mathbf{x}_k|^2 = |\mathbf{x}_1|^2 + \ldots + |\mathbf{x}_k|^2 + \sum_{i \neq j} (\mathbf{x}_i, \mathbf{x}_j) = k + k \cdot (k-1) \cdot \left(-\frac{1}{2k}\right) = k - \frac{k-1}{2} = \frac{k+1}{2}.$$

Finally

$$\cos(\xi \hat{,} \eta) = -\frac{k}{k+1}.$$

Now consider $\xi' = \frac{\xi}{|\xi|}$, $\eta' = \frac{\eta}{|\eta|}$. These points lie on the sphere and belong to the set D. By the cosine law the distance between them equals the quantity

$$|\xi' - \eta'| = \sqrt{|\xi'|^2 + |\eta'|^2 - 2 \cdot |\xi'| \cdot |\eta'| \cdot \cos(\xi' \hat{,} \eta')} = \sqrt{2 - 2\cos(\xi \hat{,} \eta)} = \sqrt{2 + \frac{2k}{k+1}}.$$

In the case 1 we found the diameter. Note that if $k = 1$ (i.e. in the second dimension) then $\xi' = \mathbf{x}_1$, $\eta' = \mathbf{x}_2$, i.e. in fact the diameter equals the side of T. However, if $k > 1$ the length of the side equals

$$\sqrt{\frac{4k+2}{2k}} = \sqrt{2 + \frac{1}{k}} < \sqrt{2 + \frac{2k}{k+1}}.$$

The length of the side tends to $\sqrt{2}$ with growth of k but the diameter of D tends even to 2, i.e. the diameter of the sphere.

Case 2. Consider the points $\xi = \mathbf{x}_1 + \ldots + \mathbf{x}_k \quad \eta = \mathbf{x}_{k+1} + \ldots + \mathbf{x}_{2k-1}$. Similarly to the case 1 we obtain that

$$\cos(\xi \hat{,} \eta) = -\sqrt{\frac{k-1}{k+1}}.$$

Again consider $\xi' = \frac{\xi}{|\xi|} \in D$, $\eta' = \frac{\eta}{|\eta|} \in D$, and finally

$$|\xi' - \eta'| = \sqrt{2 + 2\sqrt{\frac{k-1}{k+1}}}.$$

In the case $k = 2$ (i.e. if $d = 3$) we get

$$\sqrt{2 + 2\sqrt{\frac{k-1}{k+1}}} = \sqrt{2 + 2\sqrt{\frac{1}{3}}} = \sqrt{2 + \frac{2\sqrt{3}}{3}} = \sqrt{2 \cdot \frac{3 + \sqrt{3}}{3}} = 2\sqrt{\frac{3+\sqrt{3}}{6}}.$$

It is the same as in the problem 36! Clearly, points ξ' and η' are points C and A_1 respectively in the notations of the problem 36. Here we have such (expected) analogy.

40. By the problem 33 we can place any set Ω of diameter 1 into the ball B_1 with radius $\sqrt{\frac{3}{8}}$. Then translate this set so that the point $X \in \Omega$ appears on the boundary of the circle but Ω continues to lie in B_1. Draw a ball B_2 with radius 1 and center X. Since the diameter of Ω equals 1, B_2 covers Ω also. So the intersection $B_1 \cap B_2$ covers Ω.

41. Suppose that the center of B_2 is the highest point of B_1. Cut off the little hat from $B_1 \cap B_2$ from above by horizontal plane. Dissect the rest of $B_1 \cap B_2$ into 4 parts by the planes parallel to other coordinate planes. The diameter of each part is less than 1.

42. The regular simplex cannot be dissected into $n+1$ parts of smaller diameter.

44. A set of unit diameter can be placed into the unit cube. So it is sufficient to dissect the cube into parts of diameter less than 1. Cover the unit cube by $([\sqrt{n}]+1)^n$ small cubes whose sides are quite smaller than $\frac{1}{\sqrt{n}}$. The main diagonal of each small cube is quite smaller than 1. Therefore $f(n) \leqslant ([\sqrt{n}]+1)^n$.

45. By the solution of the problem 37 the radius of the subscribed sphere about a unit simplex in $\mathbb{R}^4$ equals $\sqrt{\frac{2}{5}}$. Similarly to the solution of the problem 33 the ball B_1 of radius $\sqrt{\frac{2}{5}}$ is a universal cover in $\mathbb{R}^4$. Similarly to the problem 40 the intersection of the ball B_1 with the unit ball (whose center lies on the boundary of B_1) is a universal cover in $\mathbb{R}^4$ also.

46. Similarly to the solution of the problem 41 cut off the small hat by horizontal plane, and dissect the rest by three other coordinate plans into 8 parts.

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