Mathematical Game Competition

You need to choose one of the three games below (the one which is most interesting for you) and to come up with a winning strategy for the 1st or the 2nd player. Your strategy must guarantee a victory regardless of the opponent's moves. Try not only to describe the player's moves but also to explain why the victory is inevitable. A solution without proper explanation does not count.

Do not rush into solving all the tasks as you need to save time and energy for other competitions. A good analysis of even a single game will be considered as a success.

1. "Making a stripe from a rectangle". The players have a paper rectangle of the size $m \times n$ ($m > 1$, $n > 1$) with a grid on it. The 1st player cuts the rectangle in two rectangles following any of the gridlines. The 2nd player does the same thing with any rectangle available. Then it is the 1st player's turn, and so forth. If a player can make a strip of the width 1 from all the rectangles available after his move, he wins. Which player can ensure victory regardless of the opponent's moves? Analyze the following two cases:

a) At least one of the numbers m and n is even;

b) Both m and n are odd.

2. "Coining money". There are two coiners who are coining money in turn. When it comes to his/her turn, each coiner creates a coin worth N dollars (in other words, he makes a large amount of such coins). Initially there are no coins. On each move a coiner may create a new coin if and only if its worth cannot be composed using other existing coins. If a coiner has to make a coin worth 1 dollar, he/she loses the game.

a) Prove that if the 1st coiner creates a coin worth 2 or 3 dollars on his first move, he will lose.

b) Should the 1st coiner start with creating a coin worth 4 dollars?

c) Should the 1st coiner start with creating a coin worth 6 dollars?

d) The 1st coiner created a coin worth 5 dollars, and the 2nd coiner created a coin worth 6 dollars. What strategy should the 1st coiner use to win?

e) The 1st coiner created a coin worth 5 dollars, and the 2nd coiner created a coin worth k dollars. Prove that the 1st coiner can create a coin worth $4k - 5$ dollars and cannot create any coin that is worth more than that.

f) Prove that the 1st coiner wins if he/she starts with a coin worth 5 dollars (hint: if the 2nd coiner creates a coin worth k dollars, the 1st coiner can create a coin worth $4k - 5$ dollars. If he/she wins, the problem is solved. If the 2nd coiner can win after creating a coin worth m dollars, the 1st coiner should not have created a coin worth $4k - 5$ dollars. But what should he/she have done?)

3. "Colonisation". The map shows cities some of which are connected with roads. There are two players who make moves in turn. When it comes to his/her move, a player can capture a city which has not been captured earlier. It is not allowed to capture a city if a road connects it to the city of the opponent. The player loses if he/she cannot make a move.
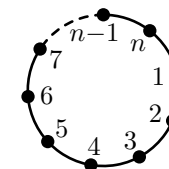
Which player can win regardless of the opponent's moves?

a) Analyze the case when they play on a map with 20 cities shown below:
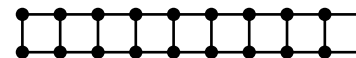
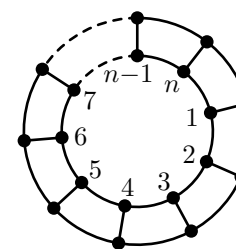b) Analyze the case when they play on a map with 20 cities shown below:
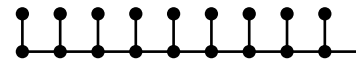
c) There are n cities located on a ring as shown below:

Which player will win depending on n?

d) There are $2n$ cities which form a double ring as shown below:

Which player will win depending on n?