Mathematical Game Competition

You need to choose one of the three games below (the one which is most interesting for you) and to come up with a winning strategy for the 1st or the 2nd player. Your strategy must guarantee a victory regardless of the opponent's moves. Try not only to describe the player's moves but also to explain why the victory is inevitable. A solution without proper explanation does not count.

Do not rush into solving all the tasks as you need to save time and energy for other competitions. A good analysis of even a single game will be considered as a success.

1. "The single neighbour". On a rectangular checked $m \times n$-board, each of two players paints a cell in turn. First cell is chosen for painting arbitrarily, and each of the subsequent ones must have a common side with some cell already painted. A cell cannot be painted twice. A player which cannot make a move loses.

Which of the players, the beginner or his rival, can win independently of moves of the other player?

Consider the cases:

a) n and m are odd integers;

b) $n = m$;

c) $n = 2$, m is arbitrary.

2. "Ones and twos". The teachers of maths put marks in the diary in turn, from left to right. Each mark is either 1 or 2. If at some moment several last figures form an integer which is a multiple of N, the teacher that made the last move loses. Which teacher, the beginner or the other player, can win independently of moves of his/her rival?

Consider the cases:

a) $N = 7$;

b) $N = 9$;

c) $N = 11$;

d) $N = 13$.

3. "Draughts on a circle". The board for the game consists of an even number of squares located along a circle. At the beginning, some consecutive squares are occupied by white draughts, and the squares opposite to them are occupied by black draughts. Some squares are free.

The players move draughts of the corresponding colours in turn. At each move a player either moves a draught to a free neighbouring square or attacks a draught of the opponent on a neighbouring cell jumping over it to the next square which must be free (at a single move, several consecutive jumps are possible). Attacked draughts are removed from the board. If a draught of the opponent can be attacked then it must be attacked, and if several draughts can be attacked then all of them must be attacked.

The aim of the game is to remove all draughts of the opponent.

If both players have made a big number of moves, for instance 1000 times as big as the number of squares and the game is not over then the draw is fixed.

If both players do their best, what is the result of the game: the win of the whites, the win of the blacks or the draw?

Consider the following cases.

a) The number of squares is 8, each player has 3 draughts.

b) The number of squares is 10, each player has 2 draughts.

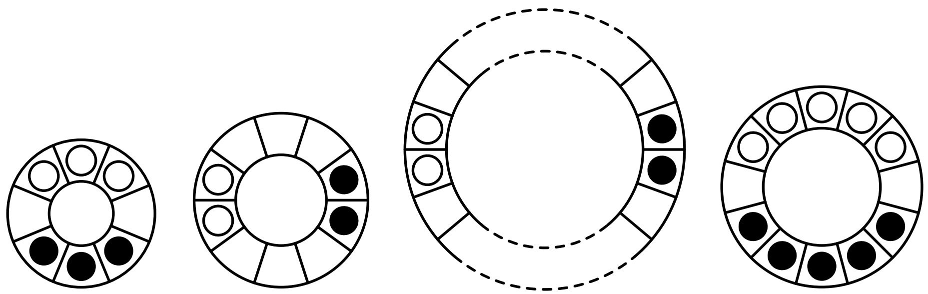
c) The number of squares is $2N$, each player has 2 draughts.

d) The number of squares is $2N$ where $N > 4$, each player has 3 draughts. Prove that in this case the blacks can guarantee the draw at worst.

e) The number of squares is 12, each player has 5 draughts. Prove that in this case the blacks also can guarantee the draw at worst.

f) The number of squares is $4N$ where $N > 2$, each player has 3 draughts. Prove that in this case both the blacks (see part d) and the whites can guarantee the draw at worst (that is, if both players do their best then the game ends in the draw).

Illustrations for the problems.



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