Solutions for «Power Sums and Bernoulli Numbers»

Here you see the solutions for some problems. Most of the solutions below were known to the jury before — but not all! Later I hope to choose the best solutions for all problems from the submitted works — that will be in fact the result of the conference. You are encouraged to ask questions (by email: `dotsenko@mccme.ru`).

1. (**a**) (**b**) The proof is standard. Let us formulate an equivalent question: how many times does the term $x_1^{a_1}\cdots x_k^{a_k}$ occur in $(x_1+\cdots+x_k)^n$? This number is equal to the number of ways to split n given brackets into k groups: first group containing a_1 brackets, second one containing a_2 of them, ... , kth one containing a_k of them. Details are left to the reader.

2. (**a**) (**b**) One can easily find a simple combinatorial interpretation for $[a]^n$ and $[a]_n$ for an integer a: for example, $[a]_n$ is the number of ways to choose a chairman, a deputy chairman, ... , a "deputy^{n-1} chairman" in a committee of a persons.

Exercise. Conclude the proof for integers a, b.

For general a, b use the following fact: a polynomial in a, b vanishes in all integer points if and only all its coefficients are equal to 0.

3. Hint: show that both $\binom{n}{k}_q$ and the coefficients of the left-hand side satisfy the same recurrence relation.

4. (**a**) The proof involves a very important method: calculus of finite differences (see [G]). The main idea is very simple: imagine that you know a polynomial $Q(x)$ such that $Q(1)=0$ ("initial condition") and $Q(x+1)-Q(x)=x^k$. Then $Q(x)=P_k(x)$: $1^k+2^k+\cdots+(n-1)^k=Q(2)-Q(1)+Q(3)-Q(2)+\cdots+Q(n)-Q(n-1)=Q(n)$. Let us show that such a polynomial of degree $k+1$ exists (and is unique). Let $Q(x)=q_0+q_1x+\cdots+q_kx^k+q_{k+1}x^{k+1}$. Then $Q(x+1)-Q(x)=(k+1)q_{k+1}x^k+(\binom{k+1}{2}q_{k+1}+kq_k)x^{k-1}+\ldots$, so it is clear that q_i can be easily found one by one from these equations (such a situation is called "a system of linear equations with an upper triangular matrix") and the $k+2$-tuple $(q_0,\ldots,q_{k+1})$ is unique (it is determined up to the "constant term" q_0, which is obtained from the "initial condition").

(**b**)

$$P_1(x)=x(x-1)/2, P_2(x)=x(x-1)(2x-1)/6, P_3(x)=x^2(x-1)^2/4,$$
$$P_4(x)=x(x-1)(2x-1)(3x^2-3x-1)/30, P_5(x)=x^2(x-1)^2(2x^2-2x-1)/12$$

5. This immediately follows from the definition of the Bernoulli numbers. Let us illustrate how powerful the method of generating functions is. Let

$$\frac{t}{e^t-1}=\sum_{k\geqslant 0}\frac{\widehat{B}_kt^k}{k!}.$$

Then it is clear that

$$t=\sum_{k\geqslant 0}\frac{t^{k+1}}{(k+1)!}\sum_{l\geqslant 0}\frac{\widehat{B}_lt^l}{l!}=\sum_{k\geqslant 0}\left(\sum_{i=0}^{k}\binom{k+1}{i}\widehat{B}_i\right)\frac{t^{k+1}}{(k+1)!},$$

so $\widehat{B}_k$ and B_k satisfy the same recurrence relation and initial condition and thus coincide. The rest of the proof is very simple: the exponential generating function for the power sums $S_k(n)$ with fixed n is

$$\sum_{k=0}^{n-1}e^{kt}=\frac{e^{nt}-1}{e^t-1}=\frac{e^{nt}-1}{t}\cdot\frac{t}{e^t-1}=\sum_{k\geqslant 0}\frac{n^{k+1}t^k}{k!}\sum_{l\geqslant 0}\frac{B_lt^l}{l!}=\sum_{k\geqslant 0}\left(\frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_in^{k+1-i}\right)\frac{t^k}{k!}.$$

6.

n	0	1	2	3	4	5	6	7	8	9
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0

n	10	11	12	13	14	15	16	17	18	19	20
B_n	$\frac{5}{66}$	0	$\frac{-691}{2730}$	0	$\frac{7}{6}$	0	$-\frac{3617}{510}$	0	$\frac{43867}{798}$	0	$-\frac{174611}{330}$

$$5,7,691,3617,43867 \text{ are prime},\ \ 6=2\cdot 3,\ \ 30=2\cdot 3\cdot 5,\ \ 42=2\cdot 3\cdot 7,\ \ 66=2\cdot 3\cdot 11,\ \ 2730=2\cdot 3\cdot 5\cdot 7\cdot 13,$$
$$510=2\cdot 3\cdot 5\cdot 17,\ 798=2\cdot 3\cdot 7\cdot 19,\ 174611=283\cdot 617,\ 330=2\cdot 3\cdot 5\cdot 11.$$

7. (This solution was suggested by E. Ferens–Sorotskij, O. Styrt and O. Evtifeeva during the conference; I did not know it before. Other participants checked that the function $f(t)=\frac{t}{e^t-1}+\frac{t}{2}$ is even, but the correct explanation meets some difficulties...)

Let k be the minimal odd number greater than 1 such that $B_k\neq 0$. $P_k(n)+n^k=P_k(n+1)=P_k(-n)$ — see solution for **9(e)** for details. This means that

$$\frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i(-n)^{k+1-i}=\frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_in^{k+1-i}+n^k.$$

By our hypothesis, $B_3 = B_5 = \cdots = B_{k-2} = 0$, therefore almost all terms cancel and we have $n^k + 2B_1 n^k + 2B_k n = 0$. Since $B_1 = -\frac{1}{2}$, we see that $B_k = 0$.

8.

$$P_1(x) = \frac{1}{2}x^2 - \frac{1}{2}x,\quad P_2(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x,\quad P_3(x) = \frac{1}{4}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2,\quad P_4(x) = \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x,$$

$$P_5(x) = \frac{1}{6}x^6 - \frac{1}{2}x^5 + \frac{5}{12}x^4 - \frac{1}{12}x^2,\quad P_6(x) = \frac{1}{7}x^7 - \frac{1}{2}x^6 + \frac{1}{2}x^5 - \frac{1}{6}x^3 + \frac{1}{42}x,\quad P_7(x) = \frac{1}{8}x^8 - \frac{1}{2}x^7 + \frac{7}{12}x^6 - \frac{7}{24}x^4 + \frac{1}{12}x^2,$$

$$P_8(x) = \frac{1}{9}x^9 - \frac{1}{2}x^8 + \frac{2}{3}x^7 - \frac{7}{15}x^5 + \frac{2}{9}x^3 - \frac{1}{30}x,\quad P_9(x) = \frac{1}{10}x^{10} - \frac{1}{2}x^9 + \frac{3}{4}x^8 - \frac{7}{10}x^6 + \frac{1}{2}x^4 - \frac{3}{20}x^2,$$

$$P_{10}(x) = \frac{1}{11}x^{11} - \frac{1}{2}x^{10} + \frac{5}{6}x^9 - x^7 + x^5 - \frac{1}{2}x^3 + \frac{5}{66}x,\quad P_{11}(x) = \frac{1}{12}x^{12} - \frac{1}{2}x^{11} + \frac{11}{12}x^{10} - \frac{11}{8}x^8 + \frac{11}{6}x^6 - \frac{11}{8}x^4 + \frac{5}{12}x^2,$$

$$P_{12}(x) = \frac{1}{13}x^{13} - \frac{1}{2}x^{12} + x^{11} - \frac{11}{6}x^9 + \frac{22}{7}x^7 - \frac{33}{10}x^5 + \frac{5}{3}x^3 - \frac{691}{2730}x,$$

$$P_{13}(x) = \frac{1}{14}x^{14} - \frac{1}{2}x^{13} + \frac{13}{12}x^{12} - \frac{143}{60}x^{10} + \frac{143}{28}x^8 - \frac{143}{20}x^6 + \frac{65}{12}x^4 - \frac{691}{420}x^2,$$

$$P_{14}(x) = \frac{1}{15}x^{15} - \frac{1}{2}x^{14} + \frac{7}{6}x^{13} - \frac{91}{30}x^{11} + \frac{143}{18}x^9 - \frac{143}{10}x^7 + \frac{91}{6}x^5 - \frac{691}{90}x^3 + \frac{7}{6}x,$$

$$P_{15}(x) = \frac{1}{16}x^{16} - \frac{1}{2}x^{15} + \frac{5}{4}x^{14} - \frac{91}{24}x^{12} + \frac{143}{12}x^{10} - \frac{429}{16}x^8 + \frac{455}{12}x^6 - \frac{691}{24}x^4 + \frac{35}{4}x^2,$$

$$P_{16}(x) = \frac{1}{17}x^{17} - \frac{1}{2}x^{16} + \frac{4}{3}x^{15} - \frac{14}{3}x^{13} + \frac{52}{3}x^{11} - \frac{143}{3}x^9 + \frac{260}{3}x^7 - \frac{1382}{15}x^5 + \frac{140}{3}x^3 - \frac{3617}{510}x,$$

$$P_{17}(x) = \frac{1}{18}x^{18} - \frac{1}{2}x^{17} + \frac{17}{12}x^{16} - \frac{17}{3}x^{14} + \frac{221}{9}x^{12} - \frac{2431}{30}x^{10} + \frac{1105}{6}x^8 - \frac{11747}{45}x^6 + \frac{595}{3}x^4 - \frac{3617}{60}x^2,$$

$$P_{18}(x) = \frac{1}{19}x^{19} - \frac{1}{2}x^{18} + \frac{3}{2}x^{17} - \frac{34}{5}x^{15} + 34x^{13} - \frac{663}{5}x^{11} + \frac{1105}{3}x^9 - \frac{23494}{35}x^7 + 714x^5 - \frac{3617}{10}x^3 + \frac{43867}{798}x,$$

$$P_{19}(x) = \frac{1}{20}x^{20} - \frac{1}{2}x^{19} + \frac{19}{12}x^{18} - \frac{323}{40}x^{16} + \frac{323}{7}x^{14} - \frac{4199}{20}x^{12} + {} + \frac{4199}{6}x^{10} - \frac{223193}{140}x^8 + 2261x^6 - \frac{68723}{40}x^4 + \frac{43867}{84}x^2$$

9. (**a**) $6P_1(x)P_2(x) = 5P_4(x) + P_2(x)$
(**b**) $3P_2(x)P_2(x) = 2P_5(x) + P_3(x)$
(**c**) $12P_2(x)P_3(x) = 7P_6(x) + 5P_4(x)$
(**d**) $2P_3(x)P_3(x) = P_7(x) + P_5(x)$
(**e**) is obvious: denote by α the leading coefficient of the given polynomial $P(x)$ of degree k and apply the inductive hypothesis to the polynomial $\widehat{P}(x) := P(x) - \alpha P_{k-1}(x)$ (of degree $k-1$).

(**f**) (This proof is, as far as I know, due to L.Positselskij. Recently I found out that the collection of formulae for $P_k(x)$ in [Pr] leads to another variant of the following lemma.)

Define an operation T on polynomials by $(TP)(x) := P(1-x)$. Assign $P_0(x) = x - \frac{1}{2}$ (!!).

Lemma. *$(TP_k)(x) = (-1)^{k+1}P_k(x)$ for all k.*

Proof. It is obvious for $k = 0$. Assume that $k > 0$. Recall that $P_k(x+1) - P_k(x) = x^k$, $P_k(1) = 0$ (see the solution for 4), $P_k(0) = 0$ (use explicit formula (1)). This means that $(-1)^{k+1}P_k(-(-x)) - (-1)^{k+1}P_k(1-(-x)) = (-x)^k$, i.e., ($y := -x$) $y^k = (-1)^{k+1}P_k(-y) - (-1)^{k+1}P_k(1-y) = (TP_k)(y+1) - (TP_k)(y)$. $(TP_k)(1) = 0$ because P_k vanishes at zero, hence the uniqueness of P_k concludes the proof. □

The rest of the proof shows the power of linear algebra. It is clear that $T(P_kP_l) = T(P_k)T(P_l) = (-1)^{k+l}P_kP_l$. Let $P_k(x)P_l(x)$ (a polynomial of degree $k+l+2$) have the following expansion in P_j's: $P_k(x)P_l(x) = \sum_{j=0}^{k+l+1} b_jP_j(x)$. It follows that $\sum_{j=0}^{k+l+1}(-1)^{k+l+1}b_jP_j(x) = (-1)^{k+l}P_k(x)P_l(x) = T(P_kP_l)(x) = \sum_{j=0}^{k+l+1}(-1)^{j+1}b_jP_j(x)$ and since such an expansion is unique we see that b_j must vanish for $j \not\equiv k+l+1(2)$.

(**g**) $60P_3(x)P_4(x) = 27P_8(x) + 35P_6(x) - 2P_4(x)$.

10. Let us introduce the notation: $a \equiv b(k)$ means that $a - b$ is divisible by k. (Most concentrated readers noticed this notation above...)

(**a**) Recall the Fundamental Theorem of Arithmetic (speaking more precisely, this statement is equivalent to the traditional form of FTA): if integers $a, b \not\equiv 0(p)$ then $ab \not\equiv 0(p)$. This implies that multiplication by a non-zero remainder modulo p is injective (i.e., maps different remainders into different ones). Therefore it is obvious that for each non-zero remainder a there exists (a unique) remainder b such that $ab - 1 \equiv 0(p)$ (there are only finitely many remainders, so the above means that each of them will occur).

(**b**) One may use the hint in order to prove the second statement. Let us deal with the first one. For each a not divisible by p the remainders of $a, 2a, \dots, (p-1)a$ form the complete system of non-zero remainders modulo p. Hence $1 \cdot 2 \cdots (p-1)(a^{p-1} - 1) = a \cdot 2a \cdots (p-1)a - 1 \cdot 2 \cdots (p-1) \equiv 0(p)$. Cancel $(p-1)!$ (it is relatively prime to p) to conclude the proof.

11. (**a**) is a straightforward corollary of the above statement: $a^{l(p-1)} - 1$ is divisible by $a^{p-1} - 1 \equiv 0(p) \dots$

(**b**) Consider the remainder $\widehat{k}$ of k modulo $p-1$. Then $a^k - a^{\widehat{k}} \equiv 0(p)$ for all a (it is obvious), so let us show that $S_{\widehat{k}}(p) \equiv 0(p)$. Let b be a non-zero remainder such that $b^{\widehat{k}} - 1 \not\equiv 0(p)$. Examine $b^{\widehat{k}} S_{\widehat{k}}(p)$. Since $b, 2b, \dots (p-1)b$ form the complete system of non-zero remainders, we see that $b^{\widehat{k}} S_{\widehat{k}}(p) - S_{\widehat{k}}(p) \equiv 0(p)$. To conclude the proof, proceed as above.

Definition. Let p be a prime. Let us call a rational number a p-integer if its denominator is not divisible by p.

12. (Here we follow [BSh].) Setting $n = p^r$ in (1), we see that

$$\frac{S_{2k}(p^r)}{p^r} - B_{2k} = \frac{1}{2k+1} \sum_{j=0}^{2k-1} \binom{2k+1}{j} B_j p^{r(2k-j)}.$$

If r is large enough the right-hand side (and the left-hand side as well) is a p-integer. On the other hand for $m \geqslant 1$

$$S_{2k}(p^{m+1}) = \sum_{u=0}^{p^m-1} \sum_{v=0}^{p-1} (u + vp^m)^{2k} \equiv p \sum_{u=0}^{p^m-1} u^{2k} + kp^m \sum_u u^{2k-1} \sum_v v \equiv pS_{2k}(p^m)(p^{m+1}),$$

and this implies that $\frac{S_{2k}(p^{m+1})}{p^{m+1}} - \frac{S_{2k}(p_m)}{p^m}$ is an integer. Thus $\frac{S_{2k}(p)}{p} - B_{2k}$ is a p-integer as well. One may replace $S_{2k}(p)$ by 0 for $2k \not\equiv 0(p-1)$ and $S_{2k}(p)$ by 1 for $2k \equiv 0(p-1)$ to obtain the following statement (which is just what we need):

For any prime number q $B_{2k} + \sum\limits_p \frac{1}{p}$ is a q-integer (here the summation is taken over primes p such that $2k \equiv 0(p-1)$).

Hence this (rational) number is an integer.

13. One may check that $44 \equiv 0(p-1)$ for a prime $p \Leftrightarrow p \in \{2, 3, 23\}$. Hence the denominator of B_{22} is 138 and $B_{22} + \frac{1}{2} + \frac{1}{3} + \frac{1}{23}$ is an integer $\Rightarrow B_{22} = 6193 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = 6193 - \frac{121}{138} = \frac{854513}{138}$.

14. Most of the participants used here the followig version of the Dirichlet theorem: there is infinitely many primes having remainder 1 modulo 3. It was an interesting experience for me to hear different proofs of this fact: for example, A. Entin and D. Faifman used the quadratic reciprocity law in order to prove it. Still I prefer solutions which avoid such strong machinery...

First solution.

Suppose that for any prime p such that n is divisible by p p is congruent to 1 modulo 6 (for example, take $n = 7, 7^2, 7^3 \dots$). Let $q \neq 2, 3$ be a prime divisor of B_{2n}'s denominator. It follows that $q \equiv \pm 1(6)$, so $q - 1$ is divisible either by 6 or by an integer $m \equiv -1(3)$. Both cases together with the equivalence $2n \equiv 0(q-1)$ and the assumption about prime divisors of n lead to contradiction which concludes the proof.

Second solution.

(It was known to me before, but I was glad to see it in the text submitted by E. Ferens–Sorotskij, O. Styrt and O. Evtifeeva.)

For a prime p all divisors of $2p$ are $1, 2, p, 2p$, so the denominator of B_{2p} is equal to 6 unless $2p + 1$ is a prime number. Otherwise try $p := 2p + 1$ (and $2p + 1 := 4p + 3$) etc. (There *will* occur a composite number: for example, $2^{p-1}p + 2^{p-1} - 1$ is divisible by $p \dots$)

15. (**a**) (**b**) Let us follow the solutions suggested by A. Entin & D. Faifman, O.Golberg, M. Levin (compare this to the one in [MS], which involves generating functions). The idea is to draw on the result of **16**: we will show that for each prime p the number $a^{2n}(a^{2n} - 1)B_{2n}/2n$ is a p-integer. It is obvious for p such that $2n$ is not divisible by $p - 1$ (use **12** and **16**). Suppose that $2n$ is divisible by $p - 1$; $2n = (p-1)p^k m$, where m is not divisible by p. For a divisible by p its power a^{2n} is divisible by p^{2n} (it is divisible by p^{k+1} because $2n \geqslant p^k \geqslant k + 1$, and the denominator of $B_{2n}/2n$ is divisible by exactly p^{k+1} — see **12**). For a not divisible by p the number $a^{2n} - 1$ is divisible by p^{k+1} because $2n$ is divisible by $p^{k+1} - p^k = \varphi(p^{k+1})$. (Here we use Euler's theorem: for a relatively prime to n there holds an equivalence $a^{\varphi(n)} \equiv 1(n)$, where $\varphi(n)$ denotes the number of integers less than n and relatively prime to n. The proof is similar to the proof of **10(b)**.)

16. (This solution was suggested by A. Entin & D. Faifman. The main advantage of this approach is that one may easily rewrite it for **17**. O. Golberg and M. Levin used the existence of a primitive root modulo p^k (for $p \neq 2$);

E. Ferens–Sorotskii, O. Styrt and O. Evtifeeva deduced it from **15**, which was deduced from a very complicated solution of **17**.)

Let $1 \leqslant g \leqslant p-1$ be such that $g^k - 1$ is not divisible by p for $0 < k < p-1$(«g is a primitive root modulo p». **Exercise.** Prove that there exists such a g — it is not very difficult.)

$$\frac{gt}{e^{gt}-1} - \frac{t}{e^t-1} = \sum_{k\geqslant 1} (g^k-1) B_k \frac{t^k}{k!} = t \sum_{k\geqslant 1} G_k \frac{t^{k-1}}{(k-1)!},$$

where $G_k = (g^k-1)\frac{B_k}{k}$. $\sum\limits_{k\geqslant 1} G_k \frac{t^{k-1}}{(k-1)!} = \frac{g}{e^{gt}-1} - \frac{1}{e^t-1}$.

Denote $e^t - 1$ by u.

$$\frac{g}{e^{gt}-1} - \frac{1}{e^t-1} = \frac{g}{(1+u)^g - 1} - \frac{1}{u} = \frac{g}{gu + \sum\limits_{j=2}^{g} c_j u^j} - \frac{1}{u} = \frac{1}{u}\left(\frac{1}{1 + \sum\limits_{j=2}^{g} c_j u^{j-1}/g} - 1 \right).$$

Let us notice that

$$\frac{1}{1 + \sum\limits_{j=2}^{g} c_j u^{j-1}/g} = 1 + \sum_{j\leqslant 1} d_j u^j,$$

where the denominators of d_j are not divisible by p. Therefore

$$\sum_{k\geqslant 1} G_k \frac{t^{k-1}}{(k-1)!} = \sum_{r\geqslant 0} a_r (e^t-1)^r = \sum_{r\geqslant 0} a_r \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} e^{mt},$$

where all a_r are p-integers. Replacing all e^{mt} by the corresponding power series gives us

$$G_k = x_1 + x_2 \cdot 2^{k-1} + \ldots x_k \cdot (k-1)^{k-1},$$

where x_k are p-integers. (Here we use the fact that the power series expansion for $(e^t-1)^m$ starts from t^m.) Therefore G_k is a p-integer for all k. Let us return to the solution. B_{2n}/n is a p-integer $\Rightarrow$ B_{2n} is a p-integer (it is obvious). Suppose that B_{2n} is a p-integer. This means that $2n$ is not divisible by $p-1$, so $g^{2n}-1$ is not divisible by p and thus B_{2n}/n is a p-integer.

17. (Most of the work was done in the above solution for **16**.) Write

$$G_{2k} = \sum_{m=1}^{2k+p-2} x_m m^{2k-1},$$

where x_m are p-integers (expand the sum $\sum\limits_{r=0}^{2k+p-2} a_r (e^t-1)^r$.). Then it is clear that

$$G_{2k+p-1} = \sum m = 1^{2k+p-2} x_m m^{2k+p-2}$$

(the proof is similar to the one above). Thus

$$G_{2k+p-1} - G_{2k} = \sum_{m=1}^{2k+p-2} x_m m^{2k-1}(m^{p-1}-1),$$

so its numerator is divisible by p.

$$G_{2k+p-1} - G_{2k} = g^{2k}(g^{p-1}-1)\frac{B_{2k+p-1}}{2k+p-1} + (g^{2k}-1)\left(\frac{B_{2k+p-1}}{2k+p-1} - \frac{B_{2k}}{2k}\right),$$

and since the numerator of $g^{2k}(g^{p-1}-1)\frac{B_{2k+p-1}}{2k+p-1}$ is divisible by p (use **10** and **12**) and $g^{2k}-1$ is not divisible by p ($2k$ is not divisible by $p-1$), we see that the numerator of $(\frac{B_{2k+p-1}}{2k+p-1} - \frac{B_{2k}}{2k})$ is divisible by p. Now one may obtain Kummer's congruence simply by induction.

18. Suppose that there are only finitely many irregular primes: $p_1, \dots, p_k$. Choose an integer n large enough to have $\left|\frac{B_{2m}}{2m}\right| > 1$ ($m = n(p_1 - 1)\cdots(p_k - 1)$). It is clear that the denominator of $\frac{B_{2m}}{2m}$ is divisible by p_i for all $i = 1, \dots, k$ (use **12** and **16**), so its numerator is equal to ± 1 (by our hypothesis, if it is divisible by a prime p, then p is one of p_i's). This contradicts $\left|\frac{B_{2m}}{2m}\right| > 1$.

19. Denote this constant term by $C(a_1, \dots, a_k)$. Let us show that $C(a_1, \dots, a_k) = C(a_1 - 1, a_2, \dots, a_k) + \cdots + C(a_1, \cdots, a_k - 1)$ (there are obvious initial conditions $C(a_1, \dots, a_{k-1}, 0) = C(a_1, \dots, a_{k-1})$ and $C(0) = 1$). Since multinomial coefficients satisfy the same recurrence relation and initial condition, the Dyson's conjecture follows. The proof is very simple: consider the polynomial $f(x) = \sum\limits_{j=1}^{n} \prod\limits_{i\neq j} \frac{x_i - x}{x_i - x_j}$. Its degree is $n-1$ and $f(x_1) = \cdots = f(x_n) = 1$, so $f(x) \equiv 1$. Thus

$$1 = f(0) = \sum_{j=1}^{n} \prod_{i\neq j} \frac{x_i}{x_i - x_j} = \sum_{j=1}^{n} \prod_{i\neq j} \frac{1}{1 - \dfrac{x_j}{x_i}}.$$

Multiply $\prod\limits_{1\leqslant i\neq j\leqslant n} \left(1 - \frac{x_j}{x_i}\right)^{a_j}$ by this equality to obtain the above recurrence relation.

21. Hint: check that the sequences $y_i(n) := \sum\limits_{k\equiv i(p-1)} \binom{2n+1}{k}$ have periods modulo p for all $i = 0, \dots p-2$.

22. Hint: this recurrence relation means that the exponential generating function for the Bernoulli numbers satisfies a certain differential equation (write out this equation!).

23. This is simply a straightforward calculation.

24. Adding up the equalities from the previous problem, we obtain

$$\sum_{0<2j<2k} \zeta(2i)\zeta(2k-2i) = \sum_{m,n} f(m,n) - \sum_{m>n} f(m,n) - \sum_{m<n} f(m,n) = \sum_{n} f(n,n) = \frac{2k+1}{2}\zeta(2k).$$

Use the recurence relation from **22** to obtain (simply by induction) the classical result

$$\zeta(2k) = (-1)^{k-1} 2^{2k-1} B_{2k} \frac{\pi^{2k}}{(2k)!}.$$

References

[BSh] Z. I. Borevich, I. R. Shafarevich. Number theory. (In Russian: Moscow, Nauka Publishers, 1985.)

[G] A. O. Gel'fond. Calculus of finite differences. (In Russian: Moscow-Leningrad, GITTL, 1952.)

[GKP] R. L. Graham, D. E. Knuth, O. Patashnik. Concrete mathematics. (In Russian: Moscow, Mir Publishers, 1998.)

[MS] J. W. Milnor, J. D. Stasheff. Characteristic classes. (In Russian: Volgograd, Platon Publishers, 1998.)

[Pr] V. V. Prasolov. Problems and theorems of linear algebra. (In Russian: Moscow, Nauka Publishers, 1996.)