Power Sums and Bernoulli Numbers

A Short Preface/Disclaimer

The text below shows the connection between two series of problems. First of them is to find an explicit formula for the sum of kth powers of the first n integers. The answer here involves one remarkable sequence of rational numbers — Bernoulli numbers. The second series of problems is about number-theoretic properties of these numbers. This sequence is often used by mathematicians who specialize in number theory, calculus (of real and p-adic numbers), algebraic topology, Lie groups etc. Therefore the most remarkable properties of this sequence are already discovered and there is not much hope for something new. Thus the problems given do not contain some exciting statements which are so new that the jury is not sure whether they are true etc. So, if you want to deal with «problems for explorers», you should certainly turn to another problem.

I use the following notation: the problems which involve strong machinery which is not well known (such as generating functions), are marked by a diamond ($^\diamond$). Standard statements (some of them help you to get acquainted with the language involved, some are just hints) are marked by a box ($^\square$). I think that it is very useful to all participants to prove these statements themselves (they are not very difficult).

Binoms: a Reminder

Some problems below involve binomial coefficients which show up in Newton's binomial theorem. That is why here you see some statements generalizing this famous result.

Definition/reminder. *Binomial coefficients $\binom{n}{k}$ are defined for $n,\, k \geq 0$ by the equality $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In general, one can define a multinomial coefficient $\binom{n}{a_1,\, a_2,\, \ldots,\, a_m}$ by $\binom{n}{a_1,\, a_2,\, \ldots,\, a_m} = \frac{n!}{a_1!a_2!\ldots a_m!}$ (for m-tuples of nonnegative integers $(a_1,\, a_2,\, \ldots,\, a_m)$ such that $a_1+a_2+\ldots+a_m = n$; it is clear that $\binom{n}{k} = \binom{n}{k,\, n-k}$). A q-binomial coefficient $\binom{n}{k}_q = \frac{n!_q}{k!_q(n-k)!_q}$, where $n!_q = 1\cdot(1+q)\cdots(1+q+\ldots+q^{n-1})$, is another generalization for this notion. Define Pochhammer symbols as follows: $[a]_n = a(a-1)\cdots(a-n+1)$, $[a]^n = a(a+1)\cdots(a+n-1)$; $[a]_0 = [a]^0 = 1$.*

1$^\square$. Prove the following identities for all $x_1,\, x_2,\, \ldots,\, x_m$:

(**a**) (I. Newton's binomial theorem.) $(x_1+x_2)^n = x_1^n + nx_1^{n-1}x_2 + \ldots + \binom{n}{i}x_1^i x_2^{n-i} + \ldots + x_2^n = \sum_{i=0}^n \binom{n}{i}x_1^i x_2^{n-i}$.

(**b**) (Multinomial theorem.) $(x_1+x_2+\ldots+x_m)^n = \sum_{a_1,\, a_2,\ldots,\, a_m} \binom{n}{a_1,\, a_2,\ldots,\, a_m} x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$, where the summation is taken over all multinomial coefficients with $a_1+\ldots+a_m = n$.

2$^\square$. Prove the following identities for all a, b and integer $n \geq 0$:

(**a**) $[a+b]_n = \sum_{i=0}^n \binom{n}{i}[a]_i[b]_{n-i}$.

(**b**) $[a+b]^n = \sum_{i=0}^n \binom{n}{i}[a]^i[b]^{n-i}$.

3$^\square$. Prove for all q, z and integer $n \geq 1$ the identity $(1+z)(1+qz)\cdots(1+q^{n-1}z) = \sum_{k=0}^n \binom{n}{k}_q z^k q^{k(k-1)/2}$. (Hence $\binom{n}{k}_q$ for all n, k is a polynomial in q, which might not be obvious at a glance.)

A question (*it is even less obligatory than other problems*)**.** Can you suggest other interpretations for $\binom{n}{k}$, $\binom{n}{a_1,\, a_2,\, \ldots,\, a_n}$, $\binom{n}{k}_q$?

Power Sums

People who know history of mathematics say that Bernoulli numbers were introduced by Jacob Bernoulli in his book «Ars Conjectandi»[1], which contains the following problem:

JB problem. *Denote by $S_k(n)$ the sum $1^k + 2^k + \ldots + (n-1)^k$. Find a polynomial $P_k(x)$ such that $P_k(n) = S_k(n)$ for all n.*

4. (**a**) Prove that for all k there exists such a polynomial $P_k(x)$.

(**b**) Find $P_k(x)$ for $k = 1, 2, 3, 4, 5$.

Bernoulli found out (when he wrote down a list of P_k's for small k) that coefficients of the polynomials P_k satisfy the following rule: there exists a sequence of numbers $B_0,\, B_1,\, B_2,\, \ldots,\, B_m, \ldots$ such that

$$S_k(n) = \frac{1}{k+1}\left(B_0 n^{k+1} + \binom{k+1}{1}B_1 n^k + \ldots + \binom{k+1}{k}B_k n\right) = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i n^{k+1-i}. \qquad (1)$$

The proof is not very difficult if the problem is well formulated (such a situation is quite typical).

[1] i.e., «The Art of Conjecturing». It is one of the first books on probability theory.

Definition. *Bernoulli numbers B_m are defined from the recurrence relation* [2]

$$\sum_{i=0}^{k}\binom{k+1}{i}B_i = B_0 + \binom{k+1}{1}B_1 + \ldots + \binom{k+1}{i}B_i + \ldots + (k+1)B_k = 0 \qquad (2)$$

for $k > 0$ and the initial condition $B_0 = 1$.

5. Prove that (1) holds for B_k defined by (2).

6. Write out the first 20 Bernoulli numbers. Decompose their numerators and denominators into primes (hint: the numerators are mostly primes, but it is not true in general — see B_{20}).

7. Prove that for odd $n > 1$ $B_n = 0$.

Now one can easily solve the JB problem.

8. Write out polynomials P_k for $k < 20$. [3]

As soon as you have written out the first 20 of P_k's, you can think of some formulae which involve these polynomials. The most obvious is $P_1(x) \cdot P_1(x) = P_3(x)$.

9. Fill the gaps: (**a**) $6P_1(x) \cdot P_2(x) = \ldots P_4(x) + \ldots P_2(x)$ (**b**) $3P_2(x) \cdot P_2(x) = \ldots P_5(x) + \ldots P_3(x)$

(**c**) $12P_2(x) \cdot P_3(x) = \ldots P_6(x) + \ldots P_4(x)$ (**d**) $2P_3(x) \cdot P_3(x) = \ldots P_7(x) + \ldots P_5(x)$

(**e**) Prove that for any polynomial of degree k there exists a unique representation $a_{-1} + a_0P_0(x) + a_1P_1(x) + \ldots + a_{k-1}P_{k-1}(x)$ with some numbers $a_{-1}, a_0, \ldots, a_{k-1}$ (which depend on a polynomial).

Now you can state some conjectures. You can even prove some of them ...

(**f**) Prove that in such a representation for $P_n(x)P_m(x)$ only polynomials P_l with even $m+n+1-l$ occur with non-zero coefficients. (Hint: what is the *correct* definition for P_0?)

... but not all of them ...

(**g**) Find such a representation for $P_3(x)P_4(x)$.

Numerators and Denominators

The following topics in number theory became popular after E. Kummer's works in 1850's, who discovered the connection between the prime divisors of Bernoulli numbers' numerators and the Last Fermat Theorem[4]. It seems to be one of the main reasons for a very strange behaviour of those numerators, although there is an explicit formula for the denominators.

10$^\square$. Let p be a prime number. (**a**) Prove that remainders modulo p form a field. (The definitions of addition and multiplication are obvious. So, you have to prove that for any a which is not divisible by p there exists b such that ab has remainder 1 modulo p and that different b's satisfying that have equal remainders modulo p.). (**b**) Prove that for any a which is not divisible by p $a^{p-1}-1$ is divisible by p and that for $k < p-1$ there exists b which is not divisible by p such that $b^k - 1$ is not divisible by p. (Hint. A polynomial of degree k over any field has no more than k roots.)

11. Prove that: (**a**) $S_k(p)$ is divisible by p unless k is divisible by $p-1$ (**b**) otherwise $S_k(p)+1$ is divisible by p.

12. (. lausen — C. von Staudt, 1840) $B_{2n} + \sum\limits_p \frac{1}{p}$ — is integral, where the summation is taken over primes p such that $2n$ is divisible by $p-1$.

13. Without drawing on the recurrence relation calculate B_{22} which is greater than 6192 and less than 6193.

14. Prove that the denominator of B_{2n} is equal to 6 for infinitely many n.

15$^\diamond$. (**a**) Prove that for any integer n $2^{2n}(2^{2n}-1)B_{2n}/2n$ is an integer.

(**b**) Prove that for any integer a, n $a^{2n}(a^{2n}-1)B_{2n}/2n$ is an integer.

One can obtain some information (in fact, very very little) about the Bernoulli numbers' numerators and denominators from the following problem.

16. (C. von Staudt, 1845) The denominator of B_{2n}/n is divisible by a prime p if and only if the denominator of B_{2n} is divisible by p.

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[2] The most convenient form of this relation is $(B+1)^{k+1} = B^{k+1}$ where you should expand the expression in left-hand side and use the so-called evaluation map: $B^i \mapsto B_i$. And those readers who worked with generating functions can check that the exponential generating function for B_i is just $B(t) = \frac{t}{e^t-1}$.

[3] Bernoulli was proud that he managed to find $S_{10}(1000)$ « for less than half quarter of an hour».

[4] Employing very strong number-theoretic machinery, he proved that the Last Fermat Theorem is true for primes p such that for any n p does not occur in the numerator of $B_{2n}/2n$. Such primes are called regular. Nobody knows if there are infinitely many regular primes, although Jensen in 1915 proved that there are infinitely many primes which are not regular. The first non-regular prime number is 37. It occurs in the $B_{32}/32$'s numerator 7709321041217.

More problems on Power Sums and Bernoulli Numbers

Some participants say that some of the problems introduced are well known (I suppose that fans of «Concrete Mathematics» comment in this way on all problems which deal with formulae). If some problems are known to you, you may just discuss the solutions with the members of the jury responsible for the problem (Volodya Dotsenko, Vadim Bugaenko) — you don't need to write down solutions in this case.

Here you see two more problems — they were introduced during the presentation.

17. (Kummer's congruence) Prove that for integers n, k and prime p such that $2n$ is not divisible by $p-1$ and $2n-2k$ is divisible by $p-1$ the numerator of $B_{2n}/2n - B_{2k}/2k$ is divisible by p.(And so you should check only finitely many cases in order to find out if given p is regular.)

18. Prove that there are infinitely many irregular primes. (Hint. You can use the following fact: $|B_{2k}/2k| > 1$ for large k.)

Power Sums and Bernoulli Numbers: after the semifinal

More on multinomial coefficients

Nobody has ever tried to suggest interpretations for …–nomial coefficients. The two following problems show very curious interpretations.

19. (Freeman Dyson's conjecture) Let $a_1, \dots, a_n$ be nonnegative integers. Prove that the constant term in the $\prod_{1 \le i \ne j \le n} (1 - \frac{x_j}{x_i})^{a_j}$ is equal to $\binom{a_1+\ldots+a_n}{a_1,\ldots,a_n}$.

20$^\diamond$. Prove that $\binom{n}{k}_q$ is equal to the number of k-dimensional subspaces in n-dimensional space over finite field $\mathbb{F}_q$.

Do you know a solution not drawing on the «Power Sums and … »?

You are encouraged both to find an elementary solution and to suggest your own problems …

21. (suggested by O. Golberg and M. Levin) Let n be an integer and $p > 2$ be a prime. Prove that $\sum\limits_{k>0} \binom{2n+1}{k(p-1)}$ is divisible by p. (It is convenient to define $\binom{n}{k} = 0$ when $n < k$.)

ζ-function: real and p-adic

The following problems contain some results on so-called Riemann's ζ-function $\zeta(s) = \sum\limits_{n \ge 1} \dfrac{1}{n^s}$.

22. Prove the following recurrence relation ($k > 1$):

$$\sum_{i=0}^{k} \binom{2k}{2i} B_{2i} B_{2k-2i} = (1-2k) B_{2k}.$$

23. Let $f(m,n)$ be the sum $\dfrac{1}{mn^{2k-1}} + \dfrac{1}{m^{2k-1}n} + \frac{1}{2} \sum\limits_{j=2}^{2k-2} \dfrac{1}{m^j n^{2k-j}}$. Prove that

$$f(m,n) - f(m+n,n) - f(m,m+n) = \sum_{0<2j<2k} \frac{1}{m^{2j} n^{2k-2j}}.$$

24. Euler proved that $\sum_{n \ge 1} \frac{1}{n^2} = \frac{\pi^2}{6}$. Use this result to calculate $\sum_{k \ge 1} \frac{1}{n^{2k}}$ for all integers k. (The answer involves Bernoulli numbers.)

25$^\diamond$. Try to explain the following equalities:

(**a**) $1 + 2 + \ldots = -\frac{1}{12}$

(**b**) $1^2 + 2^2 + \ldots = 0$

(**c**) $1^{n-1} + 2^{n-1} + \ldots = -\frac{B_n}{n}$ for an integer $n > 1$.

26. Suppose that integers m, n and prime p are such that $2m$ is not divisible by $p-1$ and $2m-2n$ is divisible by $p^N(p-1)$ for an integer N. Prove that the numerator of $(1-p^{2m-1})\frac{B_{2m}}{2m} - (1-p^{2n-1})\frac{B_{2n}}{2n}$ is divisible by p^{N+1}. (This means that «p-adic ζ-function is continuous».)