Convex Polygons

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Problems

We say that points are *in general position* in the plane if no three of them are on a line; we say that points are *in convex position* if they form a convex polygon.

1 Single Convex Polygon

1.1 Let X be any finite set of points in general position in the plane. For A, B, $C \in X$, triangle ABC is called *even*, if the number of points of X inside ABC is even and *odd* otherwise. Let A_1, A_2, ..., A_n be points of X. Prove that if all triangles with these points as vertices have the same parity (i.e., either all are even or all are odd) then points A_1, A_2, ..., A_n are in convex position.

1.2 Let A_1, A_2, A_3, ... be any finite set of points in general position in the plane. We say that triangle $A_iA_jA_k$, $i < j < k$, is *well-oriented* if its vertices come clockwise in the above order; otherwise we say that the triangle is *maloriented*. Prove that for any positive integer n points A_{i_1}, A_{i_2}, ..., A_{i_n} are in convex position if and only if all triangles $A_{i_p}A_{i_q}A_{i_r}$, $i_p < i_q < i_r$ have the same orientation.

Let $N(n)$ be the smallest positive integer such that for any set of points in general position in the plane with at least $N(n)$ points there always exists n points forming a convex n-gon. The problem of determination of $N(n)$ or at least finding a reasonable estimates of it is known as the *Erdős-Szekeres problem*.

1.3 Prove that $N(3) = 3$, $N(4) = 5$.

1.4 Prove that if we have 2000 points in general position in the plane then we can form at least 400 disjoint convex quadrangles. Can you form more? How about 500?

1.5 Prove, drawing on results of problems 1.1, 1.2, that $N(n)$ exists for any $n > 2$.

1.6 Prove that $N(5) > 8$. In fact $N(5) = 9$, but it seems that a prove of this would be too technical.

1.7 a) Prove that $N(n)$ grows exponentially, i.e. there exists $\alpha \in \mathbb{R}$ such that $N(n) > \alpha^n$ for all n.

b) Prove that $N(n) \geq 2^{n-2} + 1$.

1.8 Open question: by the conclusion of the previous problem $N(6) \geq 17$. Is it true that $N(6) = 17$?

1.9 A number $n > 4$ points in general position in the plane are marked. Prove that one can find at least $n(n-1)(n-2)(n-3)/120$ convex polygons with vertices in marked points.

2 Empty Convex Polygons

Let $H(n)$ be the smallest positive integer such that in any set of at least $H(n)$ points in general position in the plane one can always find n points that form a convex n-gon with no other points of the set inside.

2.1 a) Prove that $H(3) = 3$.

b) Consider $n > 3$ points in general position in the plane. Is it true that there always exists a circle that has at least three of given points on its circumference and no other given points inside?

2.2 Find $H(4)$.

2.3 Prove that $H(5) \geq 10$. By the way, $H(5) = 10$.

2.4 Prove that for $n \geq 7$ there exists no $H(n)$.

2.5 a) Prove that $H(6) \geq 20$.

b) Open question: does $H(6)$ exist?

3 Other Generalizations

3-1 Other Dimensions

3.1 Prove that for all n in any big enough set of points in general position in $\mathbb{R}^d$ there always exist n points in convex position.

By $N_d(n)$ denote the smallest number of points for which the above statement holds.

3.2 Open question: is it true that $N_3(n)$ grows exponentially?

3-2 Ovals instead of points

By an *oval* we mean a closed convex bounded set of points in the plain; we consider collections of pairwise nonoverlapping ovals. We say that three ovals are non-collinear if none of them is a subset of the convex hull of the other two; more generally, we say that n ovals are in *convex position* if none of them is a subset of the convex hull of the others. We say that a set of ovals determines a (convex) k-gon if k of them are in convex position (we call them vertices) and others lie in the convex hull of these k ovals.

3.3 a) Show that for any five ovals any three of which are non-collinear some four are in convex position.

b) Suppose $m \geq 3$. Prove that if in a set Φ of ovals any $k \geq m$ ovals determine only a m-gon then Φ consists of at most $m+1$ ovals.

By $G(n)$ denote the smallest positive integer such that for any set of at least $G(n)$ ovals any three of which are non-collinear one can always find n ovals in convex position. Obviously, $G(n) \geq N(n)$. The previous problem suggests that $G(4) = N(4) = 5$. One can show that $G(5) = N(5) = 9$. Whether or not $G(n) = N(n)$ for $n \geq 4$ is an open question.

3.4 Prove that $G(n)$ is finite for all n.

3-3 Abstract Convexity

A rectangular parallelepiped in $\mathbb{R}^3$ with edges parallel to coordinate axes is called *standart*. The intersection of two standart parallelepipeds, unless empty, is standart again. Thereby the class of standart parallelepipeds is similar to that of all convex sets. The implied convex structure, however, differs a lot from the usual one. For example, the next result states that a polyhedron convex with respect to this structure cannot have too many vertices.

Consider three points x, y, $z \in \mathbb{R}^3$. We say that x lies between y and z if x belongs to any standart parallelepiped that contains both y and z.

3.5 a) Prove that out of any 17 points in $\mathbb{R}^3$ at least one lies between two others.

b) Prove that among any $2^{2^{d-1}}+1$ points in $\mathbb{R}^d$ at least one lies between two others. For $2^{2^{d-1}}$ points it is not true.

The following theorem (Erdős) fits ideologically to the context:

3.6 In any sequence of $pq+1$ different real numbers one can find an increasing subsequence of length $p+1$ or a decreasing subsequence of length $q+1$.

4

4.1 Now let us mark infinitely many points in general position in the plane. Prove that there exists a convex "infinitegon" with vertices at marked points, i.e., such a (countable) infinite subset that for all k any k points of it are in convex position.

4-1 Cups and Caps

We say that points (x_1, y_1), (x_2, y_2), ... , (x_m, y_m) form an m-cap, if

$$\frac{y_1 - y_2}{x_1 - x_2} > \frac{y_2 - y_3}{x_2 - x_3} > \cdots > \frac{y_{m-1} - y_m}{x_{m-1} - x_m}$$

(in particular this implies $x_i \neq x_j$). Similarly, points form an m-cup, if

$$\frac{y_1 - y_2}{x_1 - x_2} < \frac{y_2 - y_3}{x_2 - x_3} < \cdots < \frac{y_{m-1} - y_m}{x_{m-1} - x_m}.$$

Let $f(k, \ell)$ be the smallest positive integer such that any set X of at least $f(k, \ell)$ points in general position, x-coordinates of which are all different, always contains a k-cup or an ℓ-cap.

4.2 Prove that $f(k, \ell) \leq \binom{k+\ell-4}{k-2} + 1$.

4.3 Prove that $f(k, \ell) = \binom{k+\ell-4}{k-2} + 1$.

4.4 Prove that $N(n) \leq \binom{2n-4}{n-2} + 1$.

4.5 Prove that $N(n) \leq \binom{2n-5}{n-3} + 2$.

4-2 Indivisibility Coefficient

4.6 Do there exists two disjoint finite sets A and B in the plane such that the following two conditions hold:

(i) No three points of the set $M := A \cup B$ are on a line, and the distance between any two points of M is at least 1.

(ii) Any triangle with vertices belonging to set A contains a point from B and any triangle with vertices belonging to set B contains a point from A?

4.7 For any positive integer n find maximum possible k such that *any n points in general position in the plane can be painted in red and blue in such a way that after any k points are removed convex hulls of remaining red and blue points overlap.* Prove that the ratio k/n can be arbitrary close to $1/2$ if n is sufficiently large.

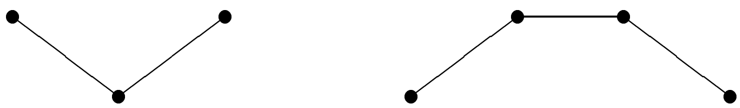

Figure 1: Examples of 3-cup and 4-cap.

Solutions

1 Single Convex Polygon

1.1. Assume the converse, then one of the points is a convex combination of three others, for example, $A_1 \in \triangle A_2A_3A_4$. We consider parity of triangle XYZ to be a modulo 2 reminder and denote it by $\mathcal{O}(XYZ)$. Then

$$\mathcal{O}(A_2A_3A_4) = \mathcal{O}(A_1A_3A_4) + \mathcal{O}(A_2A_1A_4) + \mathcal{O}(A_2A_3A_1) + 1\,,$$

where the last term corresponds to the point A_1 that lies inside triangle $A_2A_3A_4$ and contributes to its parity but does not lie inside any of the other three triangles. The equality above can not be satisfied if all four parities are equal; thus our assumption was wrong and the points are in convex position.

1.2. Assume the converse, then one of the points is a convex combination of three others, say, $A_{i_1} \in \triangle A_{i_2}A_{i_3}A_{i_4}$. Without loss of generality we may assume that triangle $A_{i_2}A_{i_3}A_{i_4}$ is well-oriented and $i_2 < i_3 < i_4$. It can be easily checked that no matter how i_1 is related to i_2, i_3, i_4, all the triangles can not have the same orientation. For example, if $i_2 < i_1 < i_3$ then triangles $A_{i_2}A_{i_1}A_{i_3}$ and $A_{i_2}A_{i_1}A_{i_4}$ have different orientation; other cases can be dealt with similarly.

1.3. See fig. 2.

1.4. (a) Choose a line not parallel to any line passing through two marked points. Draw 401 lines parallel to the chosen line in such a way that there is exactly five of our points in each of 400 strips. By 1.3 we can choose a convex quadrangle in each strip.

(b) We call a set of points in general position *splittable* if there exists a line that cuts four points in convex position off it, otherwise we call the set *nonsplittable.* To solve the problem it suffices to show that any nine-point set is splittable.

Let A, B, C be points of the given set. We call angle $\angle ABC$ empty if it contains no points of our set. Then the following lemma holds.

Lemma. *For each side AB of the convex hull of a nonsplittable set there exists a point C from this set such that both angles $\angle ABC$ and $\angle BAC$ are empty.*

Such (unique) point C will be called *the closest* point to sideAB.

Proof. Let our nonsplittable set lie to the left of us when we move from point A to point B. Take ray AB and start turning it counterclockwise around point A until it hits a point of our set which we call point D. Similarly start turning ray BA clockwise around point B until it hits a point of our set which we call point C. If points C and D coincide then this is the point we are looking for. Otherwise if there are no other points of our set inside triangle CDE (see fig. 3) then convex quadrangle $ABDC$ can be cut off the set which contradicts its nonsplittability; if there are other points of our set inside CDE we can still cut a convex quadrangle off the set; to do so, proceed as follows. Start turning ray CB counterclockwise around point C until it hits a point of our set which we call point F. Shifting line CF a little gives us a line cutting convex quadrangle $ABFC$ off our set. ☐

It is obvious that the same point can't be the closest to different sides.

Turn back to the proof of splittability of any nine-point set. By the property of splittable sets derived in the above lemma, the convex hull of a nonsplittable nine-point set can only be either triangle or quadrangle. Consider both these cases.

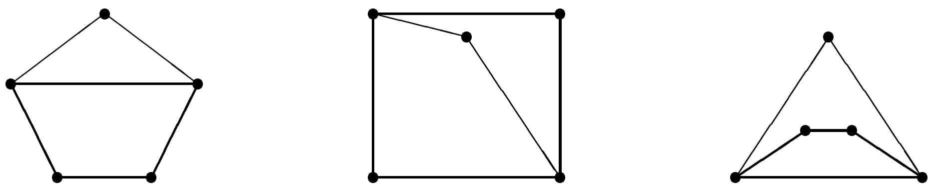

Figure 2: Any five points in general position determine a convex quadrilateral.

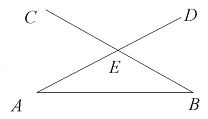

Figure 3: Construction of the closest point

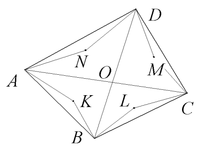

Figure 4: Quadrangle as the convex hull

Let the convex hull of our nine-point set be quadrangle $ABCD$; let K, L, M, N be the points closest to its sides and O be the intersection point of its diagonals (see fig. 4). Without loss of generality we may assume that the ninth point Z of our set lies inside triangle AOB. Then one of quadrangles $ANZK$ and $BKZL$ is convex and can be cut off our set. This contradicts nonsplittability.

Now let triangle ABC be the convex hull of our nine-point set; let points K, L, M be points closest to its sides. The other three points lie one in each triangle AKM, BKL, CLM. Indeed, if, say, triangle AKM contains no points of our set then a parallel shift of line MK away from point A cuts convex quadrangle $AK?C$ off the set. Let triangles AKM, BKL, CLM contain points P, Q, R, respectively. At least one of the segments PQ, QR, RP intersects exactly two of the segments AL, BM, CK (prove it!). Without loss of generality assume that segment PQ intersects segments BM and CK and does not intersect segment AL. Then convex quadrangle $BKPQ$ can be cut off by a line parallel to AL.

(c) No, not necessarily. Choose any 1999 points on an appropriate convex curve, such as one of the branches of a hyperbola (say, $y = 1/x$) and the last point in its center (in the origin). Then this last point forms no convex quadrangle with any three of the others.

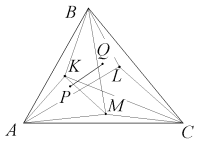

Figure 5: Triangle as the convex hull

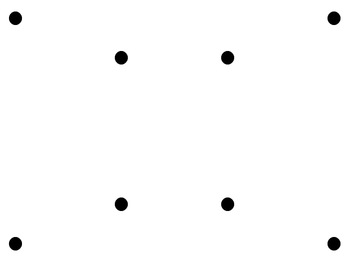

Figure 6: A set of eight points determining no convex pentagon.

(d) We have no idea.

1.5. The statement follows immediately from problems 1.1, 1.2 by Ramsey theorem. It suffices to draw on the following special case of the theorem.

Theorem (Ramsey). *For any positive integer k there exists a positive integer $R_3(k,k)$ with the following property: if there is a set of at least $R_3(k,k)$ points in general position in the plane and all triangles with vertices at these points are split into two classes then there exist k points such that all triangles with vertices in them belong to one class.*

Lower bound $N(n) \le R_3(n,n)$ is quite far from the optimal. What is known about Ramsey numbers of order three is that $2^{bn^2} \le R_3(n,n) \le 2^{c^n}$ for some constants b and c. Results of section "Cups and caps" suggest that these numbers are too large compared to $N(n)$.

1.6. Figure 6 gives an example of eight points determining no convex pentagon.

1.7. (a) F. Petrov has suggested the following construction. For every even n construct a set A_n of $2^{n/2}$ points such that no n of them are in convex position. The construction develops inductively. As the base of induction for $n = 4$ consider a set of four points that are vertices of a non-convex quadrangle. If set A_n is already constructed, choose small enough vector $\overrightarrow{v}$ not collinear with any of the lines determined by points of A_n and define $A_{n+2} = A_n \cup (A_n + \overrightarrow{v})$. It can be easily checked that set A_{n+2} contains no convex $(n+2)$-gons.

(b) We cite the proof from [4], which has its source in [2]. Construct a set X of 2^{n-2} points no n of which are vertices of a convex polygon. Let T_i, $i = 0, \dots, n-2$, be a set of $\binom{n-2}{i}$ points containing no $(i+2)$-caps or $(n-i)$-cups. Such a set exists by the result of problem 4.3 and in addition we can demand that all segments with endpoints from T_i have slopes less than or equal to one in absolute value.

For $i = 0, \dots, n-2$ place a small copy of set T_i in the neighborhood of the point of the unit circle that has angle $\frac{\pi}{4} - \frac{i\pi}{2(n-2)}$ with the real axis. Set X is then the union of these sets T_i. Then

$$|X| = \sum_{i=0}^{n-2} \binom{n-2}{i} = 2^{n-2}.$$

Assume that $Y \subset X$ is a subset of points in convex position. Let k and ℓ be maximum and minimum values of i such that $Y \cap T_i \ne \emptyset$. If $k = \ell$ then Y contains no $(k+2)$-caps and no $(n-k)$-cups. If instead $k \ne \ell$ and Y is convex then our construction guarantees that

1) $Y \cap T_k$ is a cap of at most $k+1$ points;
2) $Y \cap T_\ell$ is a cup with at most $n - \ell - 1$ points;
3) $|Y \cap T_i| \le 1$ for $i = k+1, \dots, \ell-1$.

Therefore,

$$|Y| \le k + 1 + \ell - k - 1 + n - \ell - 1.$$

We have proved that no n points of X are in convex position.

1.8. We see if anyone can solve it; we can not.

1.9. We cite the solution from [1], problem 66.

By problem 1.3, for every five points in general position in the plane one can choose four point in convex position. For each of $\binom{n}{5}$ groups of five points chosen among given n points mark four points in convex position.

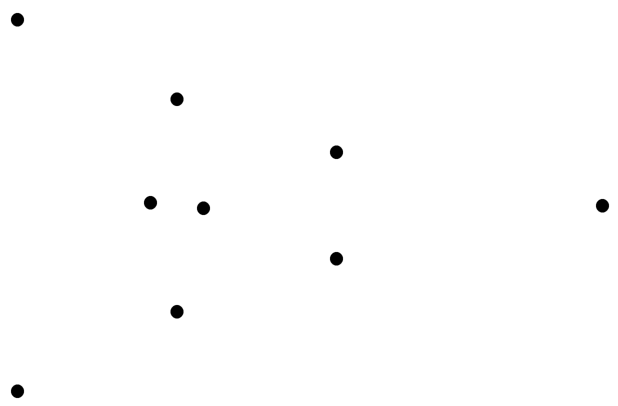

Figure 7: A set of nine points with no empty convex pentagon.

Any convex quadrangle was marked at most $n-4$ times since any four points can be completed to a set of five in $n-4$ ways only. Therefore we will end up with at least $\binom{n}{5}/(n-4)$ points; this is exactly what is stated.

2 Empty Convex Polygons

2.1. (a) It is clear for everyone who knows what triangulation means. Or consider the triangle with the smallest area with vertices in given points.

(b) We took the problem from [1], problem 99. Answer: yes. Consider the convex hull of all points; it is a polygon. Let AB be a side of this polygon. Choose the point (call it point C) for which angle ACB is maximum possible; then the circumcircle of triangle ABC is what we are looking for.

2.2. $H(4) = 5$.

2.3. Figure 7 shows a set of nine points forming no empty convex pentagon.

2.4. This solution comes from [7]. Consider only sets where x-coordinates of all points are different. By $x(A)$ denote x-coordinate of point A. We call a set of points k-closed from above for each k-cup $A_1A_2\dots A_k$ in it there exists a point A such that $x(A_1) < x(A) < x(A_k)$ and A lies above broken line $A_1A_2\dots A_k$ (polygon $AA_1A_2\dots A_k$ here need not be convex). Similarly (employing k-caps) we define sets k-closed from below. We call a set k-closed if it is k-closed both from above and from below.

For each $k \in \mathbb{N}$ construct a set X_k consisting of 2^k points that is 4-closed and has no empty convex heptagons (and such that x-coordinates of all its points are different). induction base is the one-point set.

Suppose $X = X_{k-1}$ is already constructed. Let $Y = X_{k-1} + \epsilon$ where ϵ is a vector with small enough x-coordinate and large enough y-coordinate. Then

(i) x-coordinates of points of sets X and Y alternate;

(ii) set X lies below any line connecting points of Y;

(iii) set Y lies above any line connecting points of X.

Assign $X_k = X \cup Y$.

1. Check that set X_k contains no empty convex heptagons. Assume the converse. Let Z be a convex heptagon of points from X_k. Denote $X' = A \cap Z$, $Y' = A \cap Z$. Sets X' and Y' are nonempty since neither X nor Y contains empty convex heptagons. By (ii) and (iii) X' is a cup and Y' is a cap. If set X' consists of at least four points, then by the fact that X is 4-closed from above (and again drawing on (ii), (iii)) the point of set X that makes this 4-closing lies inside heptagon Z. The case when Y' has at least four points is considered similarly.

2. Check that set X_k is 4-closed. Let $ABCD$ be a 4-cup. If $B \in Y$ then points A and C can not belong to X simultaneously by (iii). Now if, say, $C \in Y$, then by convexity all four points A, B, C, D belong to Y and because set Y is 4-closed we can find a point in Y that closes this quadruple from above. Similar considerations work if $C \in Y$. If instead both points B, C are in X then by (i) there exists a point $E \in Y$, $x(B) < x(E) < x(C)$ that

lies above segment BC and thus closing our 4-cup from above. Similarly you can check that 4-cups are closed from below.

Therefore we have constructed a family of sets X_k consisting of 2^k points and containing no empty convex heptagons. Now one can easily construct a set of n points containing no empty convex heptagons for any positive integer n. To do this it suffices to take set X_k, $2^k > n$ and remove from it one by one its extremum points until only n points are left.

2.5. (a)

3 Other Generalizations

3.1. Let us prove that $N_d(n) \leq N(n)$. In $\mathbb{R}^d$ choose a two-dimensional plane in such a way that projections of points from our set onto it do not coincide and form a set in general position. If the original set (and hence its projection) contains at least $N(n)$ points then some n points of the projection form a convex n-gon. Now corresponding points of the original set are in convex position since the property of a point "to be a convex combination of other points" is preserved under projection.

3.2. Uhu, an open question.

3.3. The statements of this and the following problems, together with the proofs are taken from [6].

(a) If a configuration of five ovals has no four ovals in convex position then any set of three, four or all five ovals in the configuration determines a triangle. That contradicts to the statement of the next point of the problem.

(b) We say that two (overlapping) ovals X and Y *form a cross* if both sets $X \setminus Y$ and $Y \setminus X$ are disconnected. If ovals do not form a cross, we say that they have simple overlapping (possibly empty). If ovals A_1, A_2, ... determine a convex k-gon and for all vertices A_i of this k-gon ovals A_i and $\operatorname{conv}(\bigcup_{j \neq i} A_j)$ have simple overlapping then we say that this k-gon is simple. For the case of a simple polygon the order of vertices is well defined.

It follows directly from the hypothesis that

$$\text{any } \ell \leq m \text{ ovals from family } \Phi \text{ determine an } \ell\text{-gon.} \tag{1}$$

Now any $\ell > m$ ovals $A_1, A_2, \ldots, A_\ell$ of family Φ form a simple m-gon M. Indeed, otherwise one of the vertex ovals, say, A_1, splits polygon M into two parts. Since the ovals are disjoint, a nonvertex oval, say, oval A_{m+1} is contained in one of the halves of polygon M which contradicts (1).

Let family Φ have $m+2$ ovals $A_1, A_2, \ldots, A_{m+2}$ that form an m-gon M with vertices $A_1, \ldots, A_m$. Denote $M_i = M \setminus A_i$, $1 \leq i \leq m$. As we have checked M_i is a simple m-gon; on of its vertices is either A_{m+1} or A_{m+2}. Since $m \geq 3$ we may assume that oval A_{m+1} is a vertex of two polygons from M_i, say, M_1 and M_j. In this case $A_{m+2} \subset M_1 \cap M_j$.

Vertices A_1 and A_j must be adjacent in polygon M, since otherwise $M_1 \cap M_2 = \operatorname{conv}(\bigcup_{i \neq 1,j} A_i)$ is an $(m-1)$-gon, even though it contains m ovals from family Φ, but this contradicts (1). Now if A_1 and A_j are adjacent vertices of polygon M, say, for $j = 2$, then

$$M_1 \cap M_2 \subset \operatorname{conv}(\bigcup_{i=3}^{m+1} A_i) \cup \operatorname{conv}(A_1 \cup A_{m+1}) \cup \operatorname{conv}(A_2 \cup A_{m+1}),$$

and A_{m+2} is contained in one of these convex hulls, which is again impossible.

3.4. Proof is the induction on n with making use of Ramsey theorem and the statement of problem 3.3. We will need the following version of Ramsey theorem:

Theorem (Ramsey). *For any positive integers n, k, ℓ there exists a positive integer $R_n(k, \ell)$ with the following property: if a set contains at least $R_n(k, \ell)$ points and all n-subsets of points are split into two classes than either there exist k points for which all n-subsets are in the first class or there exist ℓ points for which all n-subsets are in the second class.*

Induction base $n = 4$ is the statement of problem 3.3 (a). Suppose that we have already proved that $G(n-1)$ is finite. Let us prove that $G(n) \leq R_{n-1}(n+1, G(n-1)+1)$.

Consider a set Ω consisting of $R_{n-1}(n+1, G(n-1)+1)$ ovals and split all its $(n-1)$-subsets of ovals into two classes depending on whether or not these ovals determine a convex $(n-1)$-gon. Then either there exists

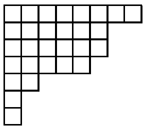

Figure 8: A Young diagram

a set Ω_1 of $n+1$ ovals any $n-1$ of which determine a $(n-1)$-gon or there exist $G(n-1)+1$ ovals no $n-1$ of which determine a $(n-1)$-gon. The latter is impossible by definition of number $G(n-1)$ and the former is impossible too if no n ovals from set Ω determine an n-gon. Indeed, in this case any n ovals of set Ω_1 determine an $(n-1)$-gon and, as it is easy to figure out, all $n+1$ ovals of set Ω_1 only determine an $(n-1)$-gon. Now the size of set Ω_1 contradicts the statement of problem 3.3 (b).

3.5. The statement of the problem follows immediately from Erdős theorem 3.6. Indeed, point $A(y_1, y_2, \ldots, y_d)$ lies between points $B(x_1, x_2, \ldots, x_d)$ and $C(z_1, z_2, \ldots, z_d)$ if for all k, $1 \le k \le d$, sequence (x_k, y_k, z_k) is monotonic.

Arrange our $2^{2^{d-1}}+1$ points in increasing order of their first coordinates. By Erdős theorem there exists a subsequence of $2^{2^{d-2}}+1$ points second coordinates of which are monotonic. Again by Erdős theorem there exists a subsequence in this subsequence with monotonic third coordinates, and so on. After applying Erdős theorem $d-1$ times we will end up with a sequence of $2^{2^0}+1=3$ points with monotonic coordinates; then the midpoint of these lies between two endpoints.

To show that the bound in the statement of the problem is exact note, in addition to Erdős theorem, that for any pair of numbers p and q one can easily construct an example of a sequence of length pq that has neither increasing subsequence of length $p+1$ no decreasing subsequence of length $q+1$. For example, split all numbers from 1 to pq onto q groups of p consecutive numbers each and arrange these groups in order of decreasing first terms. Making use of this note we can, drawing on the decomposition above, easily construct a set of $2^{2^{d-1}}$ points none of which lies between two others.

3.6. We present a well known argument employing Robinson-Shensted-Knuth algorithm. Consider our sequence. Write down its first term in the upper left square of a sheet of the grid paper. Now we add new terms and rearrange those that have been written earlier according to the following algorithm:

Step 1. Take a term of the sequence and call it current number. Call the upper left square of the sheet current square.

Step 2. If the current square is empty, fill it with the current number and switch to Step 1, otherwise switch to Step 3.

Step 3. If the current number is smaller than the number in the current square, move one square to the right, call this new square current and switch to Step 2; if current number more than that in the current square, write current number in the current square, call current the number just removed from there, move one square down, call this new square current and switch to Step 2.

After this procedure is completed terms of our sequence will be arranged in a figure called Young diagram. Young diagram consists of several horizontal rows of squares aligned along their leftmost squares where each new row contains at most as many squares as the previous one. Note that if the numbers occupy $pq+1$ squares then in the resulting diagram either the first row has more than q squares or the first column has more than p squares (otherwise the diagram would fit into rectangle $q \times p$ and would consist of at most pq squares). Both cases mean that there exists a required subsequence.

4 After the Semifinal

4.1. We need the following refinement of Ramsey theorem.

Theorem. *Let M be a countable set and k be an integer. Let all k-element subsets of M be split into two classes. Then there exists an infinite subset $L \subset M$ such that all k-element subsets of L belong to the same class.*

The proof involves induction on k and is done similarly to the usual Ramsey theorem.

In our set choose a countable subset M and enumerate its points. Split all triangles with vertices in points of M into two classes according to their orientation (as in problem 1.2). By the refined Ramsey theorem for an infinite subset $L \subset M$ all triangles with vertices from L will have the same orientation which proves the statement of the problem.

4-1 Cups and Caps

4.2. We cite the solution from [4], with the original source in [3]. The inequality follows from recurrence relation

$$f(k,\ell) \le f(k-1,\ell) + f(k,\ell-1) - 1 \tag{2}$$

and initial conditions $f(k,3) = f(3,k) = k$.

Let us prove inequality (2). Let a set X consist of $f(k-1,\ell) + f(k,\ell-1) - 1$ points. Consider all possible $(k-1)$-cups from set X. Let Y be the set of leftmost points of these cups. Then set $X \setminus Y$ has no $(k-1)$-cups. If this set contained at least $f(k-1,\ell)$ points then it would have an ℓ-cap. Suppose set $X \setminus Y$ contains less than $f(k-1,\ell)$ points, then set Y contains at least $f(k,\ell-1)$ points. If it has no k-cups then it must have an $(\ell-1)$-cap, which we denote by

$$A = \{(x_1,y_1),(x_2,y_2),\ldots,(x_{\ell-1},y_{\ell-1})\}.$$

Besides consider $(k-1)$-cup

$$U = \{(x_{j_1},y_{j_1}),(x_{j_2},y_{j_2}),\ldots,(x_{j_{\ell-1}},y_{j_{\ell-1}})\}$$

in set X for which point (x_ℓ,y_ℓ) is the leftmost point (so in our notation $j_1 = \ell-1$). It can easily be seen that either $(x_{\ell-2},y_{\ell-2}) \cup U$ is a k-cup or $A \cup (x_{j_2},y_{j_2})$ is an ℓ-cap.

4.3. The formula is obvious for $k=3$ or $\ell=3$. We prove the formula by induction. Suppose that there exists a set A of $\binom{k+\ell-5}{k-3}$ points that contains neither $(k-1)$-cups nor ℓ-caps and set B of $\binom{k+\ell-5}{k-2}$ points that contains neither k-cups nor $(\ell-1)$-caps. Consider parallel translations of these sets such that

1) set B lies to the right of set A, i.e., x-coordinate of any point of B is greater than that of any point of A;

2) the slope of each line connecting a point of A with a point of B is greater than the slope of each line connecting two points of A or two points of B.

Clearly each cup from set $A \cup B$ that contains points from both sets A and B contains at most one point from B, thus set $A \cup B$ has no k-cups. Similarly, this set has no ℓ-caps. Therefore

$$f(k,\ell) \ge \binom{k+\ell-5}{k-3} + \binom{k+\ell-5}{k-2} + 1 = \binom{k+\ell-4}{k-2} + 1.$$

4.4. This result follows immediately from the obvious inequality $N(n) \le f(n,n)$.

The statement proved above was first obtained by Erdős in [3] and for 63 years it was the best known upper bound for numbers $N(n)$. A few years ago several papers appeared almost simultaneously where slightly stronger bounds were proved, of which the strongest and still the simplest is presented in the next problem (source [5]).

4.5. Let A be an extremum point of X. Choose a point B outside the convex hull of X in such a way that no line passing through any two points of set $X \setminus A$ crosses segment AB. Through point B draw any line p that does not cross the convex hull of set X. Carry out projective transformation T that maps line p into the infinitely distant line and segment AB into a ray going straight down from point $T(A)$. Then obviously

(1) set $Y \subset X$ is in convex position if and only if set $T(Y)$ is in convex position;

(2) set $Z \subset X$ containing point A is in convex position if and only if $T(Z \setminus A)$ is a cap.

Let set X contain $\binom{2n-5}{n-3} + 2$ points. Then the set $T(X \setminus A)$ according to the statement of 4.2 contains an $(n-1)$-cap or an n-cup. Therefore X contains n points in convex position.

4-2 Indivisibility Coefficient

4.6. No, there exist no such sets. Indeed, if they do, choose any ten points from set A and let S be the set of these ten points and all points of set A that lie in their convex hull. Then since $H(5) = 10$ (see 2.3) there exist five points in set S that form a convex pentagon and having no other points of set A inside. Denote by $B_1B_2B_3B_4B_5$ such a pentagon. Let C_1, C_2, C_3 be points of set B contained in triangles $B_1B_2B_3$, $B_1B_3B_4$ and $B_1B_4B_5$, respectively. Then there exists a point from set A that lies in triangle $C_1C_2C_3$ and hence in pentagon $B_1B_2B_3B_4B_5$.

Here is a similar consideration drawing on no special facts.

Lemma. *If the hypothesis of the problem is satisfied then the convex hull of any five points of set A contains at least one other point of set A.*

Proof. If the convex hull S of five chosen points is a k-gon ($k = 3, 4, 5$), that is, if $5 - k$ of the chosen points lie inside S, then we can split this k-gon into $8 - k$ triangles with vertices at these five points (sum up the angles!). Thus there are $8 - k$ points of set B inside polygon S. Then we can find $6 - k$ nonoverlapping triangles with vertices at these points and hence there is at least one 'new' point of set A inside S. □

The statement of the problem follows immediately from the lemma. Indeed, suppose such sets exist. Fix a big enough circle, mark all points of set A inside it and choose five points with the smallest area of their convex hull (this is possible since only finitely many points fit into the circle because points are isolated from each other). By the lemma there exists one more point of A inside the convex hull of these five points. Removing one of the vertices from the convex hull of five original points gives us five points with smaller area of their convex hull. Contradiction.

4.7. Let p and s be any large enough positive integers (to be chosen later). Suppose $n = pN(2s)$. Then we can choose at least $\frac{(p-1)N(2s)}{2s}$ convex $2s$-gons from our set that have no common vertices. We color vertices of each of these $2s$-gons into red and blue alternately. It can be easily checked that one must remove at least $s - 1$ vertices from each of these $2s$-gons in order to convex hulls of its blue and red vertices not to overlap. Thus for convex hulls of red and blue points not to overlap one must remove at least $\frac{(s-1)(p-1)N(2s)}{2s}$ points. Therefore

$$\frac{k}{n} \geq \frac{\frac{(s-1)(p-1)}{2s}N(2s)}{pN(2s)} = \frac{1}{2} \cdot \frac{s-1}{s} \cdot \frac{p-1}{p}.$$

This expression can be arbitrarily close to $1/2$.

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