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Feuerbach family

Introduction.

The Feuerbach point is the touch point of the incircle and the nine-point circle. We construct a family of circles passing through the Feuerbach point which contains four remarkable circles of a triangle: the incircle, the nine-point circle, the circle passing through the feet of internal bisectors and the circle passing through the extouch points.

We propose polar correspondence as a method of solving of our problem.

Polar correspondence is one of the most refined branches of geometry. Studying of the topic requires little specific preparation. Polar correspondence helps solve many problem without writing down any formulae and even a difficult (from traditional viewpoint) problem can be done in one line.

An important object of our analysis will be a quadruple of points on a line picked in a special manner. Let point B lies on a segment AC. The segment is then split in ratio $\frac{AB}{BC}$. Find point D on line AC that splits segment AC in the same ratio "from outside", i.e., $\frac{AD}{DC} = \frac{AB}{BC}$ (fig. 1).

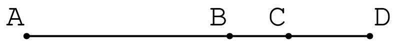

Fig. 1

Such a quadruple is called ***harmonic*** and points B and D are called ***harmonically dual*** with respect to segment AC.

If we take into account segment orientation in the above definition (so that the ratio is positive if the segments have the same direction and negative otherwise) than $\frac{AD}{DC} = -\frac{AB}{BC}$ ore

$$\frac{AD}{DC} : \frac{AB}{BC} = -1.$$

Cross-ratio $\frac{AD}{DC} : \frac{AB}{BC}$ is defined for any four points on a line, including infinitely distant one.

If, for example, B is infinitely distant then $\frac{AB}{BC} = -1$ and $\frac{AD}{DC} : \frac{AB}{BC} = -\frac{AD}{DC}$. In particular, the point dual to the middle of a segment is infinitely distant.

The cross-ratio of four points can be transferred from one line to another by means of central projection (see problem 1).

Therefore, if four rays with common origin are fixed, they cut any line at four points with the same cross-ratio no matter what line. Thus same cross-ratio is known as ***the cross-ratio of four rays***. A quadruple of rays for which if is equal to –1 is called ***harmonic***.

Polar of a Point with Respect to an Angle.

Draw two lines through a point P cutting at points A, B, C and D the sides of an angle with the vertex at point O. Let Q be the intersection of diagonals of quadrangle $ABCD$ (see fig. 2). Let us proof that rays OP and OQ form a harmonic quadruple with the sides of the angle. To do that it suffices to proof that these rays cut harmonic quadruple P, T, Q, S on line PQ.

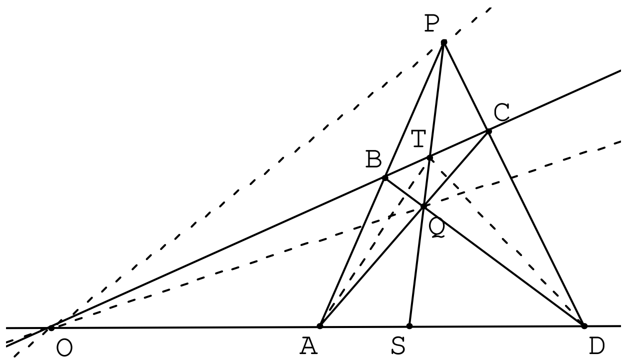

Fig. 2.

Transfer the cross-ratio of the above four points from line PQ to line AC by means of rays AP, AT, AQ, AS. We get the cross-ratio B, T, C, O ($P\rightarrow B$, $T\rightarrow T$, $Q\rightarrow C$, $S\rightarrow O$). Now transfer this ratio back to line PQ by means of rays DB, DT, DC, DO. We now get quadruple Q, T, P, S ($B\rightarrow Q$, $T\rightarrow T$, $C\rightarrow P$, $O\rightarrow S$). Note that we have been transferring the cross-ratio from one quadruple to another without changing it; hence the cross-ratio of points P, T, Q, S remains unchanged when P and Q are switched. That can only bee the case if points P and Q split segment TS in the same ratio (in absolute terns), i.e., P, T, Q and S form a harmonic quadruple.

Varying secants that we draw through point P we will find out that line OQ is the locus of points dual to P with respect to the segment of a secant that lies inside the angle. Line OQ is called the ***polar*** of point P with respect to the angle. If the point is taking inside the angle (point Q, for instance), similar construction (with AB and CD as diagonals of $ACBD$) shows that OP is the polar of point Q with respect to the same angle.

Polar of a Point with Respect to a Circle.