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Feuerbach Family

Main Problems

Notation: we mark by # a problem not directly related to the theorem and by * an open question.

9#. The A-excircle of ΔABC touches BC at point A_1 (fig. 9). Points B_1 and C_1 on AC and AB are such that lines AA_1, BB_1 and CC_1 concur. (For example B_1 and C_1 may be the other extouch points.) Let lines B_1A_1 and C_1A_1 meet the excircle secondary at points B_2 and C_2. Prove that lines B_1C_1 and B_2C_2 intersect on BC (fig. 9).

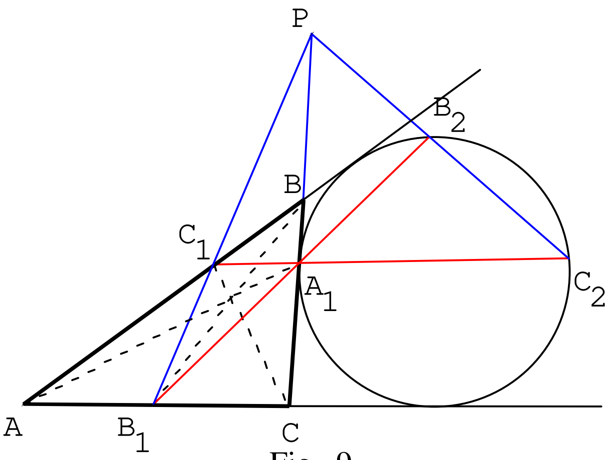

Fig. 9

10#. Let $ABCA_1B_1C_1$ be a pentahedron with two triangular faces ABC and $A_1B_1C_1$, which we call bases and three quadrangular faces, which we call side faces. On each side face mark the intersection point of its diagonals: A_0 on BB_1C_1C, B_0 on AA_1C_1C, C_0 on AA_1B_1B. Prove that the base planes and the plane that passes through points A_0, B_0 and C_0, intersect on one line, possibly infinitely distant (i.e. are parallel).

11#. Let A_0 и C_0 be the midpoints of BC and AB respectively, AA_1 the bisector of angle A, A' and B' the intouch points on BC and CA. Prove that AA_1, A_0C_0 and AB' concur (Fig. 10).

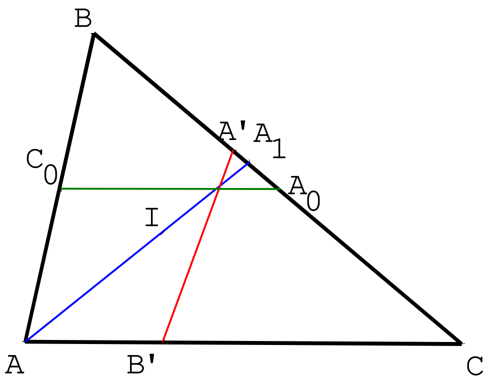

Fig. 10

12. Let H_2 and H_3 be feet of two altitudes of ΔABC, B_1 and C_1 be the feet of two internal bisectors, E and G be vertices of the intouch triangle as in figure 4. Prove that H_2H_3, B_1C_1 and EG concur (fig. 11).

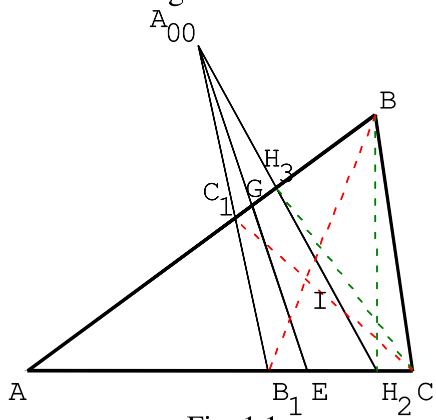

Fig. 1 1

13. Let A_1, B_1 and C_1 be the feet of the internal bisectors, D, E and G be the intouch points (fig. 12). Let E_1 be the reflection of E across BB_1, G_1 be the reflection of G across CC_1, D_1 be the reflection of D across AA_1. Prove that E_1G_1 passes through A_{00} (similarly for E_1D_1 and D_1G_1).

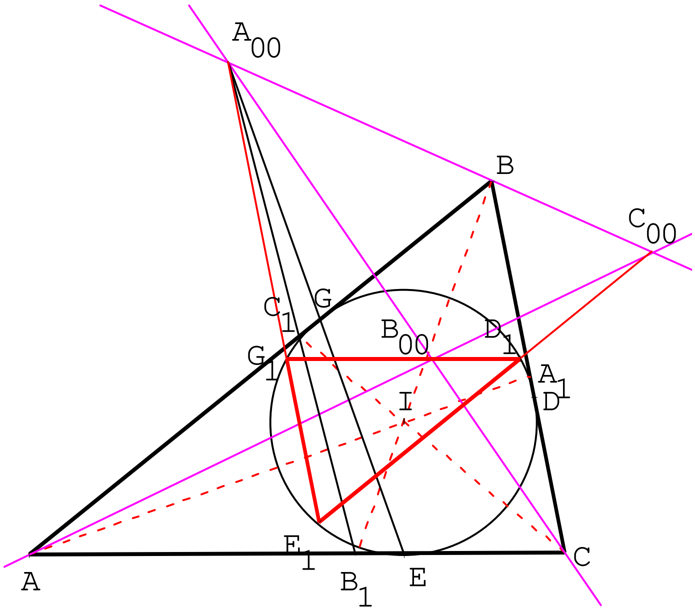

Fig. 1 2

14. Prove that ΔABC and $\Delta D_1E_1G_1$ (fig. 12) are homothetic.

15. Let A_0, B_0 and C_0 be the midpoints of BC, AC and AB, respectively. Prove that A_0A_{00}, B_0B_{00} and C_0C_{00} pass through the Feuerbach point (fig. 13).

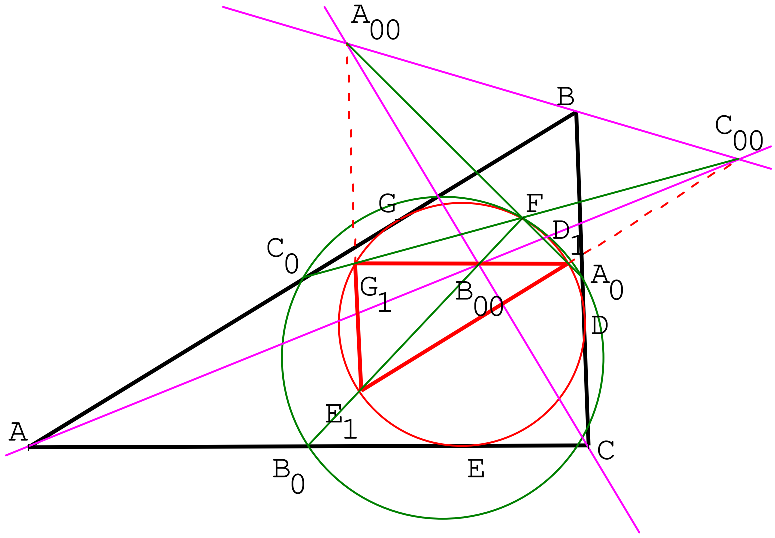

Fig. 13

16. **Theorem**. Let $H_1H_2H_3$ be the orthic triangle of ΔABC, DEG be the intouch triangle, A_{00}, B_{00}, C_{00} be the intersection points of their corresponding sidelines. Let Y be a variable point on line AC, X be the intersection point of YC_{00} and BC, Z be the intersection point of YA_{00} and AB; then

a) line XZ passes through B_{00};
b) XYZ is the chevian triangle of ΔABC;
c) circle (XYZ) contains the Feuerbach point. (Fig. 14)

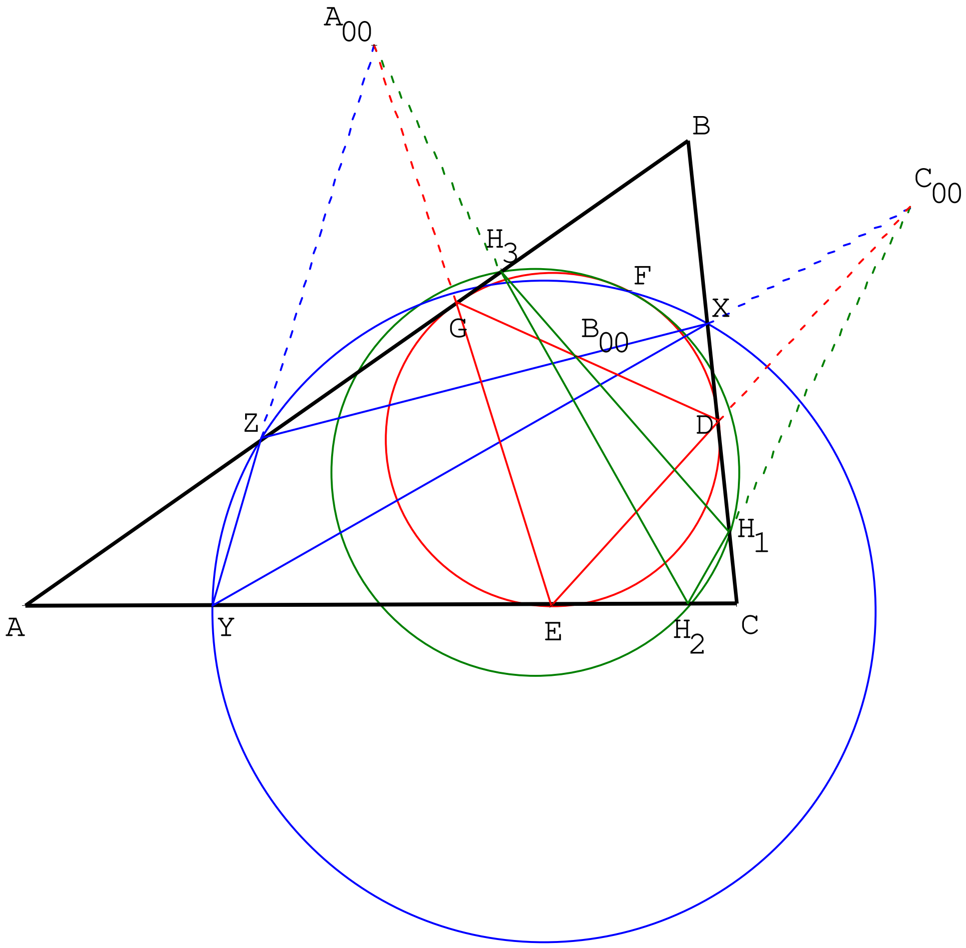

Fig. 14

17. The circle passing through the feet A_1, B_1 and C_1 of the internal bisectors of ΔABC contains Feuerbach point (fig. 15).

18. Let A_2, B_2 and C_2 be the extouch points of ΔABC on sidelines BC, AC and AB, respectively. Prove that circle $(A_2B_2C_2)$ passes through the Feuerbach point (fig. 15).

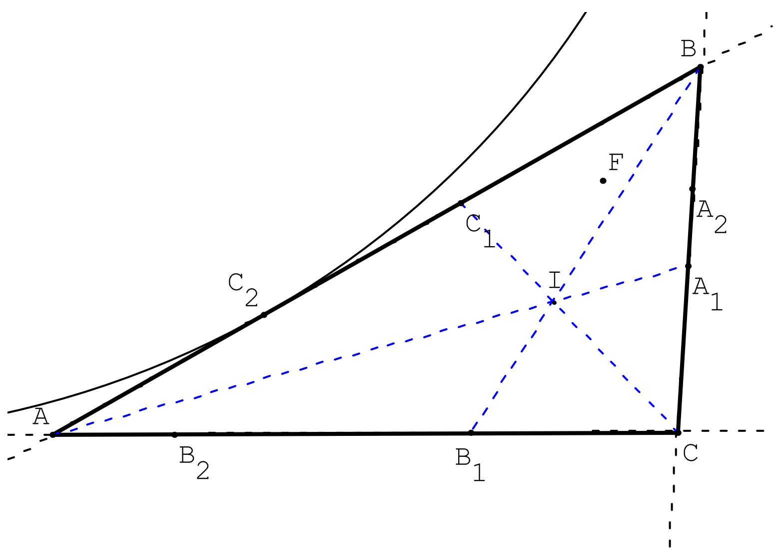

Fig. 15

19. Generate poles A', B' and C' by the intouch triangle and the medial triangle. Prove that lines AA', BB' and CC' are parallel.

20. Reflect sidelines of the orthic triangle with respect to corresponding sidelines of the intouch triangle. Prove that the resulting lines form a triangle with vertices on the incircle.

21*. Prove that triangles XYZ and $A_{00}B_{00}C_{00}$ are perspective, i.e. lines XA_{00}, YB_{00} and ZC_{00} are concurrent. Prove that the locus of such perspective centers is a line tangent to the incircle.

22*. Prove that the line connecting a vertex of the intouch triangle with the intersection point of its opposite side with corresponding sideline of the medial triangle passes through the Feuerbach point.

23*. (*Alternative definition of Feuerbach Family*)

Let $\Delta A_1B_1C_1$ be homothetic to the intouch triangle with the center at incenter I of ΔABC (fig. 16). It is known that AA_1, BB_1 and CC_1 concur. Let X, Y, Z be their intersection points with BC, AC and AB, respectively. Prove that the family of circles (XYZ) is the Feuerbach family.

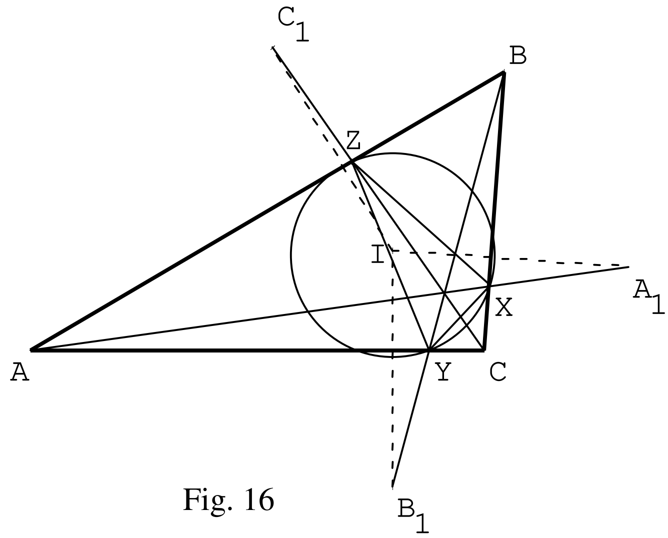

Fig. 16