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Feuerbach Family

Solutions of the Introductory Problems

0. It suffices to prove that **if point Q lies on the polar of point P then point P lies on the polar of point Q.**

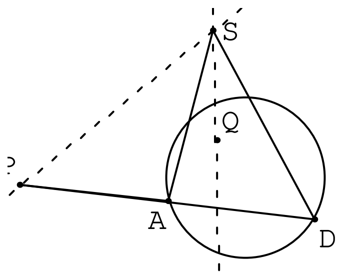

Fig. 16

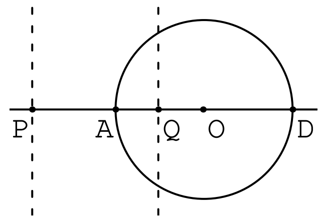

Fig. 17

Indeed, let S be the intersection point of the polars of points P and Q (fig. 16), AD be the secant passing through point P. If point Q lies on the polar of point P then SP, SA, SQ and SD are four harmonic lines; hence SP is the polar of point Q. If instead the polars of points P and Q are parallel (fig. 17) then the result is straightforward since they are orthogonal to the same diameter and split it harmonically.

1. Draw a line parallel to line A_1D_1 through point A (fig. 18). It sufficed to prove that cross-ratio of points A, B, C, D equals to that of points A, B_2, C_2, D_2. (They obviously are equal for parallel lines.) Menelaus theorem, applied to triangle ACC_2 and line BB_2, suggests

$$\frac{AB}{BC}\cdot\frac{CO}{OC_2}\cdot\frac{C_2B_2}{B_2A}=-1.$$

Similarly for DD_2

$$\frac{AD}{DC}\cdot\frac{CO}{OC_2}\cdot\frac{C_2D_2}{D_2A}=-1.$$

Equating these expressions and rearranging the terms gives

$$\frac{AB}{BC}:\frac{AD}{DC}=\frac{AB_2}{B_2C_2}:\frac{AD_2}{D_2C_2}.$$

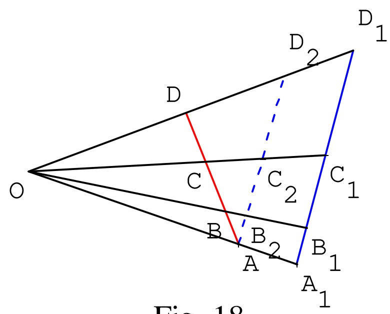

Fig. 18

That proves that cross-ratios are equal.

2. It follows from the construction that KO is the polar of point M with respect to angle AKD, therefore rays KM, KA, KO and KD form a harmonic quadruple. They cut a harmonic quadruple of points on line EF one of that is infinitely distant (since EF || KM) which implies that point O is the midpoint of segment EF.
3. It follows from the way line SQ is constructed that it is the polar of point P with respect to the circle, which means that it is orthogonal to OP, QED.
4. The polar correspondence with respect to a circle with center at point O associate parallel lines with vertices A and A_1 since they lie on the same diameter; the same applies to vertices B and B_1, as well as to C and C_1. These three pairs of parallel lines form two triangles with pairwise parallel sides that are thus homothetic. Therefore the lines connecting corresponding vertices are concurrent and their poles parallel, which completes the proof. It can be easily seen that the converse also holds.

5. By Menelaus theorem for $\triangle$ ABC and line A_1C_1 (fig. 19)

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CK}{KA}=-1,$$

and by Ceva theorem $\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}=\frac{AB_1}{B_1C}$. Hence

$$\frac{AB_1}{B_1C}:\frac{AK}{KC}=-1, \text{ QED}$$

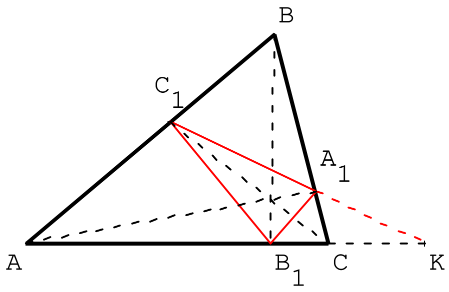

Fig. 19

6. Let lines A'C' and AC intersect at point R (fig. 20). To prove that point B lies on line A'C' it sufficed to show that cross-ratios of points A, C_2, B, C_1 and A, B_2, R, B_1 are equal to each other. By Ceva theorem

$$\frac{AC_2}{C_2B}=\frac{AB_2}{B_2C}\cdot\frac{CA_2}{A_2B},$$
$$\frac{AC_1}{C_1B}=\frac{AB_1}{B_1C}\cdot\frac{CA_1}{A_1B}.$$

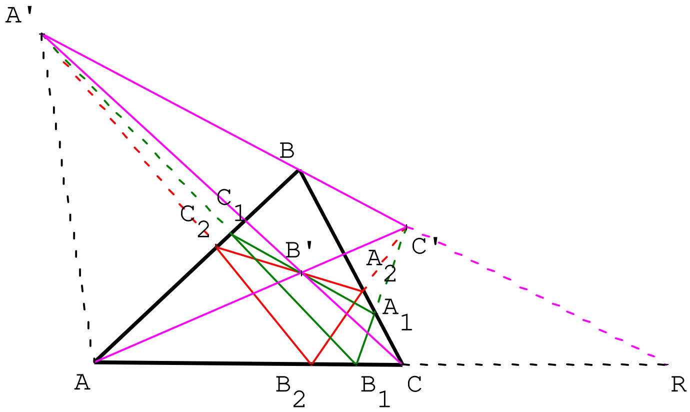

Fig. 20

Drawing on these equalities and on the fact that cross-ratio of points C, A_2, B, A_1 equals that of points C, B_2, R, B_1 we get

$$\frac{AC_2}{C_2B}:\frac{AC_1}{C_1B}=\frac{AB_2}{B_2C}\cdot\frac{CA_2}{A_2B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=\frac{AB_2}{B_2C}\cdot\frac{CB_2}{B_2R}\cdot\frac{B_1R}{B_1C}\cdot\frac{CB_1}{B_1A}=\frac{AB_2}{B_2R}:\frac{AB_1}{B_1R}.$$

That proves the desired equality.

7. Points A, B_1, C, and K form a harmonic quadruple (fig. 21). Hence so do lines B'A, B'B_1, B'C, and B'K. These lines, in turn, generate harmonic quadruple of points M, B_1, A', C_1. Therefore, line B'C' together with point A' splits segment B_1C_1 harmonically and, moreover, passes through the vertex of angle BAC. That means that line B'C' is the polar for point A'.

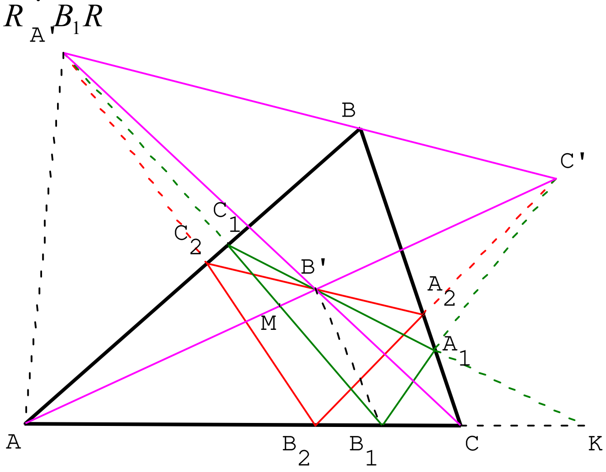

Fig. 21

8. Lines AB and AC touch the incircle at points C_1 and B_1 (fig. 22) and hence are the polars of points C_1 and B_1 with respect to the incircle; they are thus concurrent to the polar of point A_{00}. That means that point A lies on this polar. Now since line B'C' passes through point A and, besides, together with point A_{00} splits chord B_1C_1 harmonically, it is the polar of point A' with respect to the incircle.

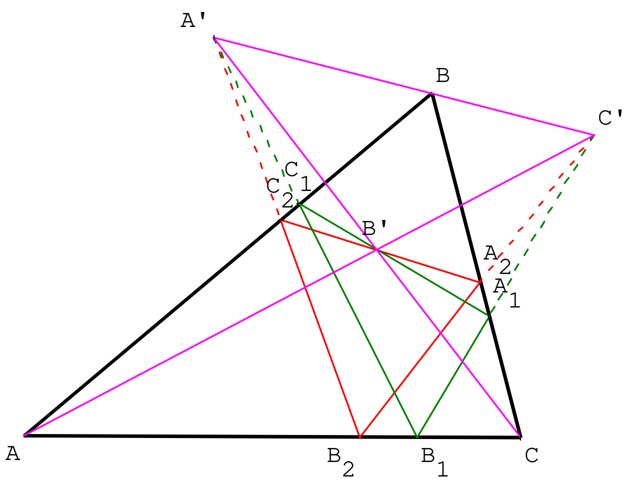

Fig. 22