Lev Emelyanov and Tatiana Emelyanova

Feuerbach Family

Solutions of the Main Problems

9. Let line MK that connects touchpoints of the circle with continued sides intersect line BC at point P (fig.23). Point A is the pole of line MK with respect to the circle and point A_1 is the pole of line BC, hence line AA_1 is the polar of point P with respect to the circle and with respect to angle BAC, since rays AP, AB, AA_1 and AC cut a harmonic quadruple of points on line PK. Since the intersection point of lines BB_1 and CC_1 lies on line AA_1, line B_1C_1 passes through point P. What is left is to show that line B_2C_2 passes through point P too. Mark intersection points D and H of line AA_1 with lines B_1C_1 and PB_2. The circle splits segment PH harmonically, but so do lines B_1A_1 and C_1A_1 (central projections of harmonic quadruple B_1, D, C_1, P). Therefore point C_2 lies on line PB_2, i.e., line B_2C_2 passes through point P.

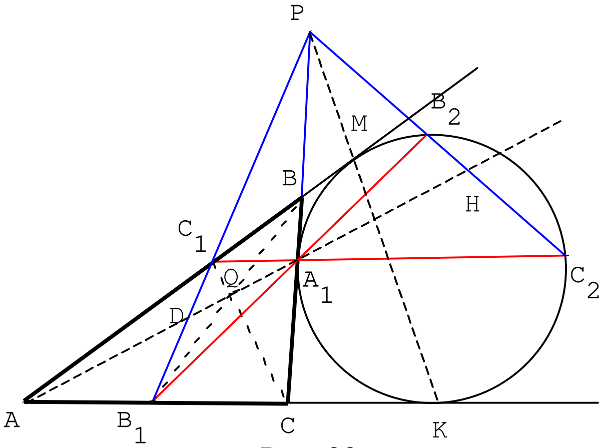

Рис. 23

10. Let the side face planes intersect on lines AA_1, BB_1 and CC_1, then these lines intersect at one point P (intersection point of the side face planes), possibly infinitely distant. That means that our pentahedron is a frustum of a tetrahedron. Consider face AA_1B_1B (fig. 24). Let lines AB and A_1B_1 intersect at point S, which, obviously, lies on the intersection line of base faces. Line SC_0 is the polar of point P with respect to angle B_1SB, hence point M where it crosses edge BB_1 splits, together with point P, this segment harmonically. Similarly for point K on edge AA_1. Considering two remaining side planes we find polars of point P in each of them. Let N be the point on edge CC_1 that is harmonically conjugent to point P with respect to this segment. The position of each of the points K, M and N on side edges is uniquely determined as harmonically conjugent to point P. Therefore at each of them intersect two polars from adjacent faces. Thus all three polars lie in the same plane – that of triangle KMN (which is also that of triangle $A_0B_0C_0$) and each of them passes through a point on the intersection line of the base planes. Then this plane itself passes through the intersection line of the base planes.

<u>*Remark.*</u> Plane $A_0B_0C_0$ has properties of a polar but with respect to a dihedral angle. Such plane is called ***polar plane.***

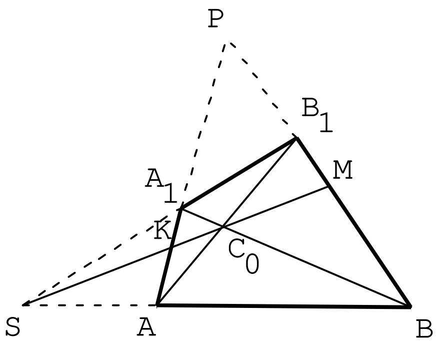

Fig. 24

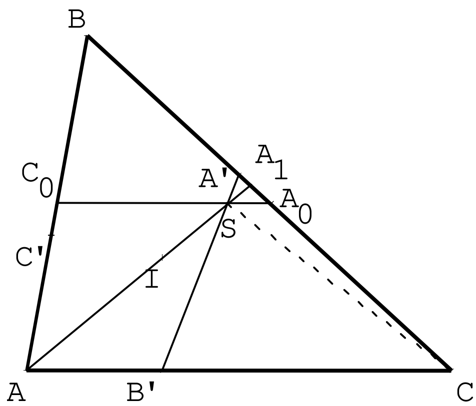

Fig. 2 5

11. We parametrise the triangle by tangent segments (fig. 25): $AB' = AC' = x$, $BC' = BA' = y$, $CB' = CA' = z$. Let us find the cross-ratio of points C, A_0, A_1, A'. It can easily be calculated that $CA_0 = \dfrac{y+z}{2}$, $A_0A_1 = \dfrac{(z-y)(z+y)}{2(2x+y+z)}$, $A_1A' = \dfrac{x(z-x)}{2x+y+z}$. Hence the required cross-ratio is equal to

$(C, A_0, A_1, A') = -\frac{x}{z}$. Now consider the quadruple of points C, A_∞, A, B' on line AC (A_∞ is the intersection point of lines AC and A_0C_0). Since point A_∞ splits segment AC in ratio –1, and point B' splits it in ratio $\frac{x}{z}$, the cross-ratio of points (C, A_∞, A, B') is also equal to $-\frac{x}{z}$. If we denote by S the intersection poin of lines AA_1 and $A'B'$, then it turns out that line A_0A_∞, which coincides with line A_0C_0, passes through the intersection points of three lines AA_1, $A'B'$ and CS.

12. Let us show that cross-ratios of quadruples A, C_1, G, H_3 and A, B_1, E, H_2 are the same. Similarly to the previous problem this will imply that lines $A_{00}A$, C_1B_1, GE, H_1H_2 are concurrent. Suppose $AB = c$, $BC = a$, $CA = b$, then (fig. 11) $AH_2 = c \cdot \cos A$, $AE = \frac{b+c-a}{2}$, $AB_1 = \frac{b \cdot c}{c+a}$. Now let us compute the lengths of the segments for cross-ratio (A, B_1, E, H_2): $B_1E = AE - AB_1 = \frac{(c-a)(c+a-b)}{2(c+a)}$, $EH_2 = AH_2 - AE = \frac{2c \cdot \cos A - b - c + a}{2}$. After rearrangements we get $\frac{AB_1}{B_1E} \cdot \frac{EH_2}{H_2A} = -\frac{c^2 - a^2 - bc + ab}{(c-a)(c+a-b)\cos A} = -\frac{1}{\cos A}$. Hence the cross-ratio of the indicated points on side AC is determined only by the size of angle A, which is the angle between sides AC and AB, and hence will be the same for corresponding points on side AB.

13. The other tangent to the incircle from point B_1 passes through point E_1 (fig. 26), so line EE_1 is the polar of point B_1 with respect to the incircle. Ssimilarly line GG_1 is the polar of point C_1. Since points B_1, C_1 and A_{00} lie on the same line (problem 6), lines EE_1 and GG_1 cross on line $B_{00}C_{00}$. And once line EG passes through point A_{00}, so does line E_1G_1.

14. Let us prove, for example, that $E_1G_1 \parallel BC$. Let R and T be the intersection points of lines E_1B_1 and G_1C_1 with line BC. $\angle BRB_1 = \angle BAB_1 = \angle CTC_1$ (fig. 26). Hence $G_1T = TD = RD = RE_1$ as tangent segments forming pairwise equal angles. So E_1G_1TR is a trapezoid and $E_1G_1 \parallel BC$.

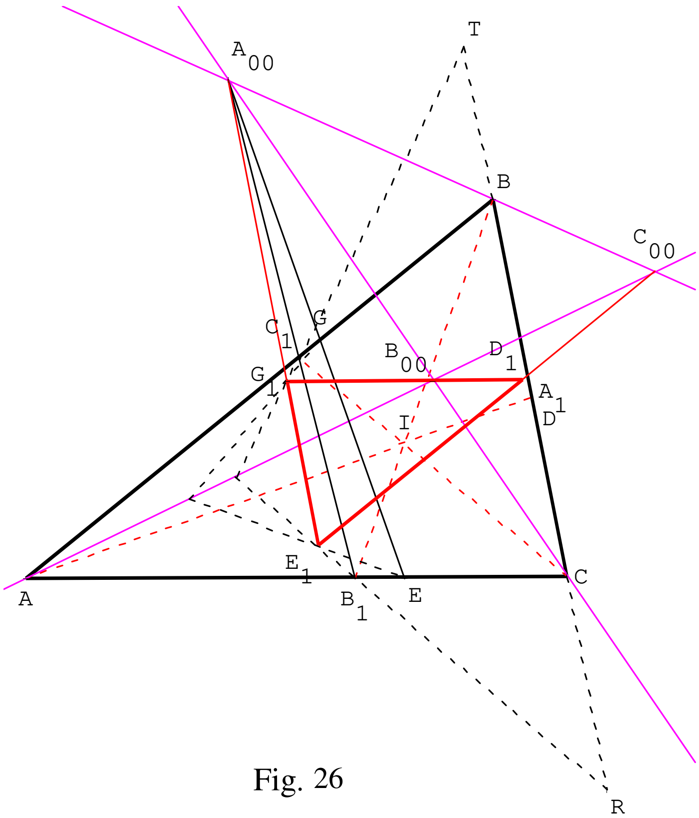

Fig. 26

15. Line $A_{00}E_1$ crosses the incircle at points D_1 and E_1 and it crosses line $B_{00}C_{00}$ at point S. A_{00}, D_1, S, E_1 – is a harmonic quadruple of points. Hence lines $B_{00}A_{00}$, $B_{00}D_1$, $B_{00}S$ and $B_{00}E_1$ form a harmonic quadruple too. One of these lines ($B_{00}D_1$) is parallel to side AC (problem 14) and, therefore, three other lines cut equal segments on line AC, i.e., line $B_{00}E_1$ passes through the midpoint of segment AC, which is point B_0 (fig. 27). Similarly, line $A_{00}G_1$ passes through point A_0, and line $C_{00}D_1$ passes through point C_0. Triangles

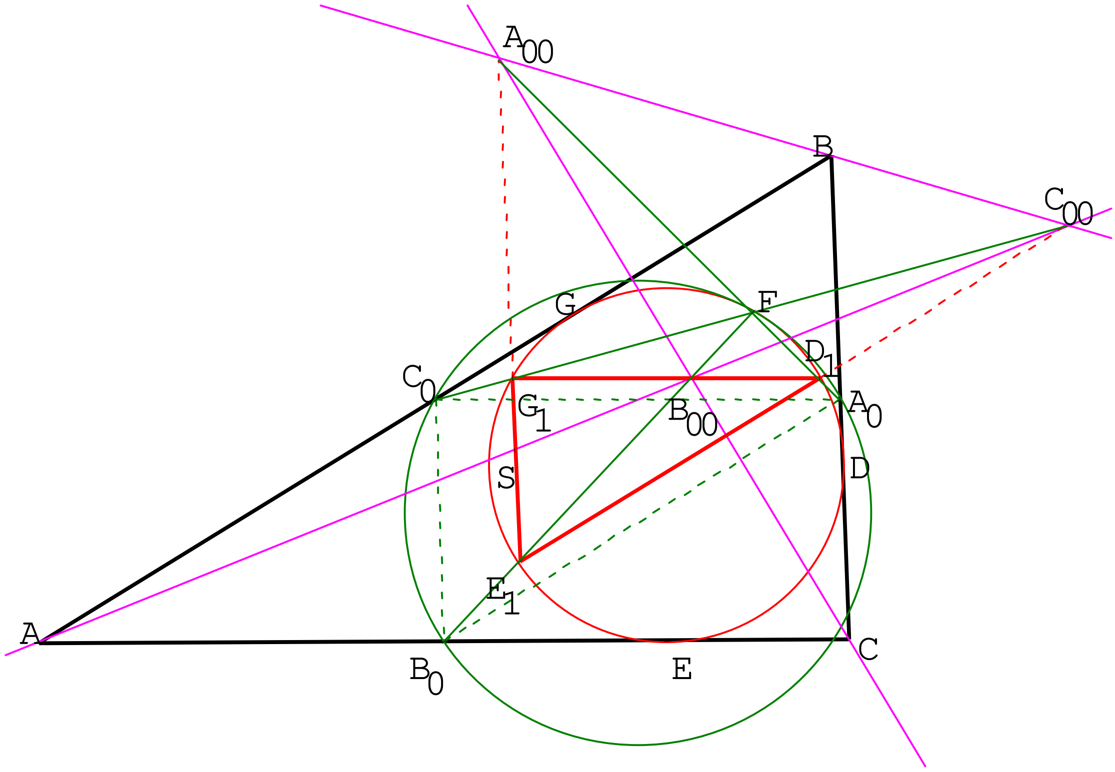

Fig. 27

$G_1E_1D_1$ and ABC are homothetic with negative coefficient (problem 14), median triangles of triangles $A_0B_0C_0$ and ABC are again homothetic with negative coefficient. Consequently, triangles $G_1E_1D_1$ and $A_0B_0C_0$ are homothetic with positive coefficient, i.e., are homothetic with center at point F where corresponding circumcircles (the incircle and the nine-point circle of ΔABC) touch. Therefore lines $A_{00}G_1$, $B_{00}E_1$ and $C_{00}D_1$ pass through the Feuerbach point, QED.

16. We prove points (a) and (b) for general case when pole forming triangles are any cevian triangles $A_1B_1C_1$ and $A_2B_2C_2$.

(a) Denote by U and V intersection points of B'A' with AB, and B'C' with BC respectively (fig. 28). Construction of points X, Y and Z implies that cross-ratio of points A, Z, C_1, U equals that of points V, X, A_1, C, hence line ZX passes through point B'.

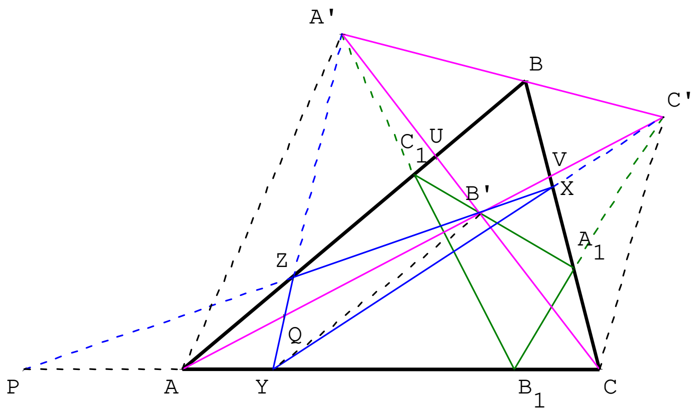

Fig. 28

(b) Let Q be the intersection point of lines YA' and AB', P be the intersection point of lines AC and ZX. A', Z, Q, Y is a harmonic quadruple since A' is a pole and AB' is the corresponding polar with respect to angle BAC. Thus lines B'A', B'Z, B'A, B'Y form a harmonic quadruple and since they cut line AC at points C, P, A, Y, respectively, these latter form a harmonic quadruple too. Therefore X, Y, Z are cevian feet (problem 5).

To prove point (c) of the theorem it suffices to show that the family of circles circumscribed around ΔXYZ, generated by moving point Y in a way described in the theorem has the only fixed point. This point will be the Feuerbach point since the latter lies on two circles of the family – the incircle and the nine-point circle. Our immediate goal is to show that this family contains three lines, i.e., moving circle 'straightens' thrice and these three lines pass through the Feuerbach point.

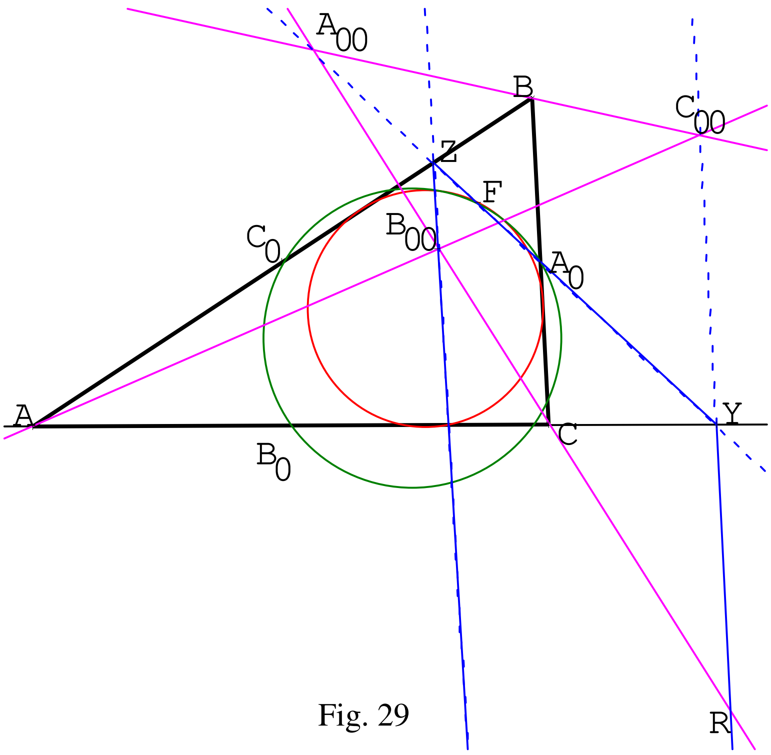

Fig. 29

Let us show, for example, that line $A_{00}A_0$ is a 'circle' from our family. Indeed, if one of the cevian feet, say, point X, becomes the infinitely distant point of line BC then sides ZX and YX turn into lines parallel to BC (fig. 29). Let R be the intersection point of lines XY and $A_{00}B_0$. Since points C_{00}, Y, R, X form a harmonic quadruple and X is an infinitely distant point, we have $C_{00}Y$ = YR. But segment $C_{00}R$ is parallel to segment BC. Thus line $A_{00}Y$ splits in halves segment BC too, and therefore, coincides with line $A_{00}A_0$. The circle passing through points Y, Z and infinitely distant point X, in turn, has infinite radius, i.e., degenerates into line YZ (in our case it coincides with line $A_{00}A_0$).

Therefore the Feuerbach family is completed with three lines A_0A_{00}, B_0B_{00} and C_0C_{00} (with lines connecting poles with midpoints of the sides).

Introduce coordinate system (x, y) in such a way that real axis coincide with line AC. Let point Y have coordinates $(t, 0)$. Then coordinates of points Z and X look like $\left(\frac{p_1 t + q_1}{a_1 t + b_1}, \frac{r_1 t + s_1}{a_1 t + b_1}\right)$ and $\left(\frac{p_2 t + q_2}{a_2 t + b_2}, \frac{r_2 t + s_2}{a_2 t + b_2}\right)$, i.e., are linear-fractional functions of t (which can easily be seen). Let us show that Feuerbach family is a one-parameter family defined by a third-degree equation. To do this we need to write down the equation of the circle passing through points X, Y and Z with coordinates defined above. Determinants are the most compact way to do it.

As it is known, the equation of the circle passing through three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by equation $\Delta(x, y) = 0$, where

$$\Delta(x, y) = \begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}.$$

Decomposing the determinant over the first row rearranges the equation to

$$s \cdot (x^2 + y^2) + a \cdot x + b \cdot y + c = 0,$$

where $s = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ which is geometrically interpreted as the doubled oriented area of the triangle with vertices at (x_1, y_1), (x_2, y_2), (x_3, y_3).

Determine what coefficients s, a, b, c look like when coordinates of points Y, Z and X are substituted. For example, $s=\begin{vmatrix} t & 0 & 1 \\ \dfrac{p_1t+q_1}{a_1t+b_1} & \dfrac{r_1t+s_1}{a_1t+b_1} & 1 \\ \dfrac{p_2t+q_2}{a_2t+b_2} & \dfrac{r_2t+s_2}{a_2t+b_2} & 1 \end{vmatrix}=\dfrac{P_3(t)}{(a_1t+b_1)(a_2t+b_2)}$, where $P_3(t)$ is a third degree polynomial of t.

Denominator roots, as well as $t=\infty$, turn s into infinity. This corresponds to infinite area of $\triangle$XYZ when one of its vertices becomes infinitely distant. Roots of the numerator turn s into zero. That happens when two vertices of $\triangle$XYZ turn into one of the vertices of $\triangle$ABC (the determinant for s has two identical rows).

Computing similarly coefficients a, b and c, we can rewrite our equation as

$$\frac{P_3(t)}{(a_1t+b_1)(a_2t+b_2)}\left(x^2+y^2\right)+\frac{P_3(t)Q_3(t)}{(a_1t+b_1)^2(a_2t+b_2)^2}x+$$
$$+\frac{P_3(t)R_3(t)}{(a_1t+b_1)^2(a_2t+b_2)^2}y+\frac{P_3(t)S_3(t)}{(a_1t+b_1)^2(a_2t+b_2)^2}=0,$$

where $Q_3(t)$, $R_3(t)$ and $S_3(t)$ are third degree polynomials of t and polynomial $P_3(t)$ is a common divisor of all coefficients since they all vanish when two of three points X, Y, Z coincide (there are two identical rows in determinant Δ).

Multiply both sides of the equation by $\dfrac{(a_1t+b_1)^2(a_2t+b_2)^2}{P_3(t)}$ and show that the resulting equation $(a_1t+b_1)(a_2t+b_2)(x^2+y^2)+Q_3(t)x+R_3(t)y+S_3(t)=0$ encompasses the entire family of circles except for one line (namely line $B_{00}B_0$).

Let us see what happens to the circle passing through points X, Y, Z when point Y, moving along line AC, hits point A, i.e., coincides with point Z (fig. 30) and polynomial $P_3(t)$ vanishes. In this case the circle can not be generally determined by its two points. But the continuity of coefficients of the last equation helps conclude that for the appropriate value of t it will again be an equation of a circle while secant ZY turns into a tangent. Therefore it is the circle passing through point X and touching line AA_{00}. Furthermore, for two finite values of t when coefficient at x^2+y^2 vanishes this equation determines lines $A_{00}A_0$ and $C_{00}C_0$. (line $B_{00}B_0$ is the limiting line for our family at $t=\infty$.)

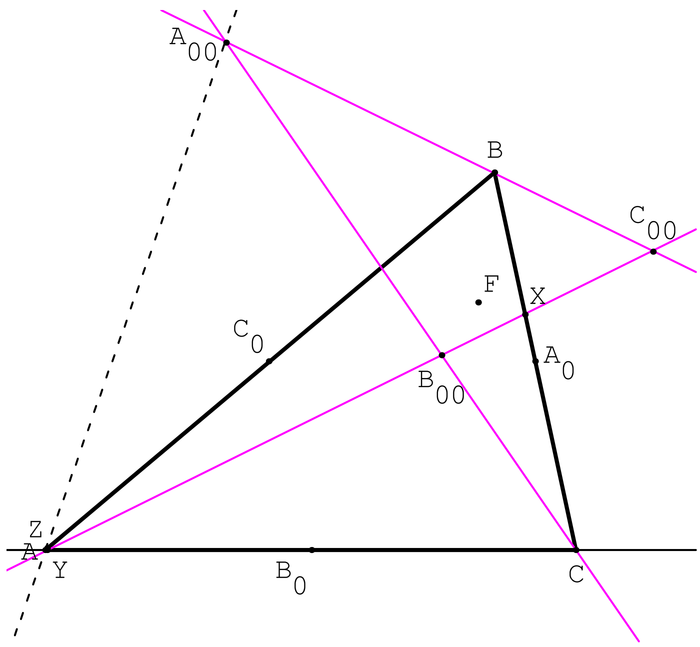

Fig. 30

These lines, together with the incircle and the nine-point circle, are four family members corresponding to different value of t pass through the same point – Feuerbach point. Substituting coordinates of this point into our equation we get an equation on t that has at most three different roots. This implies that the Feuerbach point is a fixed point of this family (and the unique fixed point as it is readily seen). This completes the proof of the theorem.

17. It follows from problem 12 that the line passing through two bisector feet passes also through the corresponding pole of the triangle defined in the theorem (problem 16). Now it follows from the theorem that the circle passing through bisector feet is contained in the Feuerbach family.

18. If in problem 12 we replace bisector feet B_1 and C_1 by touchpoints B_2 and C_2 of exscribed circles then it can easily be shown that cross-ratios of points A, C_2, G, H_3 and A, B_2, E, H_2 are again the same. Therefore the circle passing through the touchpoints of the exscribed circles is contained in the Feuerbach family.

19. Let A', B', C' be the poles generated by the intouch triangle and the midpoint triangle. Let us prove, for example, that AA' || BB'. Line B'C' is the polar of point A' with respect to angle BAC, hence AA', AB, AC', AC form a harmonic quadruple of lines. Three of them – AB, AC', AC – cut equal segments on line BB' (B' lies on the midpoint line). Thus AA' || BB'.

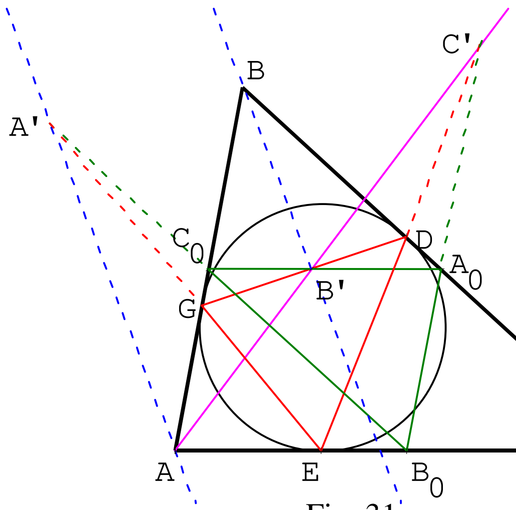

Fig. 31

20. Let H_2H_3 be a side of the orthic triangle, EG be a side of the intouch triangle as on figure 11. Line H_2H_3 is antiparallel to BC. Reflecting line H_2H_3 with respect to line EG gives a line parallel to BC and passing through intersection point A_{00} of lines H_2H_3 and EG. We have already encountered such a line in problem 13 – it is line E_1G_1. Hence $\Delta D_1E_1G_1$ (fig. 12) with vertices on the incircle is what we need.