The Hunting on the Snark
(solutions)

0. (by A. Golubov and M. Kalenkov)

Informally speaking the idea of the solution can be described as follows. First the Baker chooses a large enough square centered at the initial position of the Snark. The Baker tries to lock the Snark within this large square. During first several moves Baker fills with the cakes four squares in the corners of the large square. These four smaller squares should also be large enough. Then, while the Snark is "far enough" from the border of the large square the Baker forms a net on this border with relatively small gaps. When the Snark reaches the boundary layer of the border the Baker fills in the gaps. The boundary layer is the band of width 9.

It appears that the perimeter of the large square is eight times as big as the number of moves that the Snark needs to reach the border of this square. Since $t/8 > t/9 + \mathrm{const}$ for large enough t the Baker is able to form a net with 8-square gaps (in other words, a net of density $1/9$) and fill the corner squares with cakes (sides of the corner squares should be greater than the width of the boundary layer) before the Snark reaches the boundary layer. This way the boundary layer is split into four different parts — four rectangles going along the border of the large square. Right after the Snark jumps into the boundary layer the Baker starts to fill in the gaps in the net in the following way. He tries to build a barrier of three cakes underneath the Snark. Once such a barrier is built the Baker can easily expand this barrier on time ever after no matter which direction the Snark moves. Now we need only to prove that the Baker can create a barrier before the Snark escapes from the large square.

When the Snark jumps into the boundary layer for the first time the Baker puts a cake at the projection of the Snark on the nearest side of the large square ("underneath him"). As the Snark moves the Baker puts cakes underneath him, therefore there are always two cakes near the projection of the Snark. Should the Snark change the direction along the border or jump directly towards it the Baker immediately puts the third cake near the projection of the Snark and his goal is fulfilled. If the Snark makes eight moves in the same direction, he stands above a cake of the net at some moment. At this moment the Baker puts the third cake to the barrier. Once the barrier has been created, the Baker can expand it on time as the Snark moves around it. So when the Snark moves to the border of the square, he runs into the barrier formed by three cakes.

ε**.** It is obvious that if we fix a horizontal line L at some distance above the Snark's current position then there exists a segment on this line through which the Snark will necessarily pass (reachable area). In one move the Snark can change his x-coordinate at most by n while his y-coordinate increases at least by 1, therefore his reachable area can be covered by a the segment of length $\ell = 3x(n+1)$, where x equals the number of horizontal lines between the current position of the Snark and line L. In what follows Below we will call this segment the reachable area. It is sufficient to build a barrier of height n above the reachable area in order to prevent the Snark from crossing line L.

The Snark cannot reach distance $x/2$ from line L in less than $\frac{x}{2n}$ moves. Meanwhile, the Baker partially builds the barrier over the reachable area by placing cakes uniformly on the line next to line L from below. He places $\frac{x}{2n}$ cakes while the whole barrier requires $3x(n+1)\cdot n$. So the Baker can fill

$$\frac{x}{2n^2 \cdot 3x(n+1)} > \frac{1}{6n^3}$$

share of the barrier at least.

Consider the moment when the Snark passes through half distance x, then the width of his reachable area is halved (as ℓ is a linear function of x). Therefore the number of cakes needed for building the barrier is halved too. While the Snark is passing through half of the remaining distance $x/2$, the Baker can fill another $1/6n^3$ share of the barrier over his updated reachable area. After $6n^3$ repetitions of the above procedure the Baker will complete the barrier over the reachable area of the Snark. If x is large enough, e. g. $x > 2^{6n^3+1}$, this the Baker will succeed in catching the Snark.

2ε**.** This is a corollary of the next problem.

3ε**.** Let D be equal to $1\,000\,000 + 1$. We will show that the Baker can force the Snark to increase its y-coordinate at least once in T moves. Then this Snark can be caught by using the strategy of catching the ε-Snark of velocity Tn.

Now we describe the strategy of pushing the Snark upwards. As long as the Snark does not go above the position he holds at the beginning of pushing (*starting position*), the Baker builds walls of height D and width n to the left and to the right of the starting position. There should be $4Dn^2 + 1$ vertical lines between these walls and their upper borders should lie on the same horizontal line that the Snark was at the beginning of pushing. After completing the walls the Baker fills the space between them. In this way the Snark will be forced to increase his y-coordinate at least once in $(4Dn^2+1)D+2Dn$ moves. So T is equal to $(4Dn^2+1)D+2Dn$.

4ε**.** This is a corollary of the next problem.

5ε**.** Let D be equal to $1\,000\,000+1$. Let N denote the maximum number of moves required to catch the 3ε-Snark of the same velocity for the same value of D. The Baker chooses so huge a square centered at the origin that this square contains the initial position of the Snark. The side of this square we will denote by $2K+1$ where the value of K will be indicated below.

On the first stage the Baker fills the four corner squares in the huge square with cakes. Their sides will be $Nn+\max(D,n)$, then we can take $K=4n(Nn+\max(D,n))^2+Nn+\max(D,n)$. While the Snark is moving further than Nn squares from the border of the huge square, the Baker fills squares in the huge square as he likes. But sooner or later the Snark will be forced to come closer than that. From that moment the distance between the origin and the Snark depends on one of his coordinates only. This gives the Baker an opportunity to catch the Snark as in problem 3ε.

1/2001. Suppose the Baker can catch the suspicious Snark. Then there exists a number k such that the Snark will be caught in less than $k+1$ moves. We suppose that the Baker puts cakes according to his strategy, i. e., his behavior is deterministic. Let us consider the *tree of the game*. It consists of all possible positions of the game after the Baker's moves. An edge connects one position to another iff the latter can be obtained from the former after the Snark's move and the Baker's answer. Hence each vertex of the tree has only finite number of outgoing edges (no greater than $(2n+1)^2$). The Snark is caught in any case, therefore there are no infinite branches in the tree. It follows that the tree has only finite number of vertices. It can be easily proved by induction. If the tree would have infinite number of vertices we would be able to construct an infinite branch. On each step we would choose a branch going from the last chosen vertex having infinitely many descendants (such a branch exists due to finite branching at any vertex).

Now the Baker will use the following strategy to catch the usual Snark. If the Snark jumps to a square that he couldn't reach before, the Baker plays according to the strategy for catching the suspicious Snark. But if the Snark jumps to a square that he could jump on beforehand, the Baker "forgets" all the moves from the moment the Snark could jump on this square for the first time. Then the Baker continues to catch the suspicious Snark. It may happen that the Baker should put a cake on the square already occupied with a cake. In this case he can put a cake on any square inside a large square centered at the initial position of the Snark with side $2nk+1$.

If the Snark behaves as a suspicious one during k moves he will be caught. Otherwise he will never leave the large square and therefore he will be caught too.

1/2000. Let us enumerate all squares of the plane.

It is clear that the strategy described in problem 5ε can be generalized to the case of any square considered as an origin and any number D. Therefore the Baker can do every 2nd move according to the strategy for catching 5ε-Snark of velocity $2n$ for $D=D(x_1,y_1)$ (considering (x_1,y_1) as the origin). Every 2nd move of the remaining ones the Baker will perform the same strategy against the 5ε-Snark of velocity $4n$ for $D=D(x_2,y_2)$ (considering (x_2,y_2) as the origin) etc. The Snark will be forced to come closer to each of the squares.