Preparatory Problems. Keys and Solutions

0.1. Superwords $W = \dots vvhvv \dots$ from letters v with the only letter h certainly does not belong to class S. Another example of such superword W: $\dots vvvhhh \dots$ (first letters v only, then letters h only).

0.2.1. For lines L with finite slope k_L superwords W_L are periodic if and only if k_L is rational.

For $k_L = \infty$, i.e., for vertical line L superword W_L is periodic too since it consists of letters h only.

0.2.2. If line L has slope $k_L < 1$ then letters v occur more often then h in superword W_L.

0.2.3. Substitute $h \rightleftarrows v$ in any superword W_L from S. The resulting superword W' remains in class S since $W' = W_{L'}$ for line L' with slope $k_{L'} = 1/k_L$.

0.3. Superword $W = \{\dots h\,v\,v\,v\,h\,v\,v\,h \dots\} \in S$ because all the squares of polygon M_w corresponding to word $w = \{h\,v\,v\,v\,h\,v\,v\,h\}$ can be crossed by a line (for example, with slope $k_L = 3/7$).

0.3.1. Between any two *consecutive* letters h in W can be *either two or three* letters v. Let us prove it. If a superword $W_L \in S$ contains subword hv^kh (k letters v between two consecutive h) then slope k_L is, clearly, confined between $1/(k+1)$ and $1/(k-1)$.

The word $hvvvhvvh$ contains subwords hv^kh for $k = 2$ and $k = 3$. Therefore $k_L \in [1/3, 1/2]$.

Moreover, $k_L \neq 1/3$ and $k_L \neq 1/2]$ (why?). Hence $k_L \in (1/3, 1/2)$; but this implies $2 \le k \le 3$ for any subword $hv^kh \in W$, QED.

0.3.2. Let us list all different subwords of length 2, 3 and 4 in W:

3 words of length 2: hv, vv, vh; 4 words of length 3: vvv, hvv, vhv, vvh;

5 words of length 4: $hvvv, vhvv, vvhv, vvvh, hvvh$.

0.3.3. Superwords W_L and W_{L^*} contain subword $hvvvhvvh$, for example, for parallel lines L and L^* with slope $k_L = 3/7$; slopes $k_L = 3/8$ and $k_L = 2/5$ will work too (check it). Same applies to k_L close enough to the above values.

0.3.4. In view of 0.3.1, superword W in 0.3 is $\{\dots v\,v\,h\,v\,v\,v\,h\,v\,v\,h\,v\,v \dots\}$. Therefore, W contains the following 6 subwords of length 5: $hvvvh, vhvvv, vvhvv, vvvhv, vhvvh, hvvhv$ and 7 subwords of length 6: $vhvvvh, hvvvhv, vvhvvv, vvvhvv, vvhvvh, vhvvhv, hvvhvv$.

Since between any two consecutive letters h in W are *al least two and at most three* letters v, any subword W of length 5 or 6 contains either one or two letters h. Simple check proves that above listed are *all* subwords of lengths 5 and 6 in W.

Therefore, for any two lines L and L^* crossing all squares of polygon M_w corresponding to word $w = \{h\,v\,v\,v\,h\,v\,v\,h\}$, sets of all five-letter and six-letter subwords of W_L and W_{L^*} necessarily coincide.

On the contrary, sets of all seven-letter subwords in W_L and W_{L^*} need not coincide for all L and L^*. For example, let L and L^* have slopes $k_L = 3/7$ and $k_{L^*} = 3/8$, respectively. In this case W_L are W_{L^*} are periodic superwords $\{h\,v\,v\,v\,h\,v\,v\,h\,v\,v\}$ and $\{h\,v\,v\,v\,h\,v\,v\,v\,h\,v\,v\}$, so W_L contains subword $\{h\,v\,v\,h\,v\,v\,h\}$, which W_{L^*} misses, and W_{L^*} contains subword $\{v\,v\,v\,h\,v\,v\,v\}$, which W_L misses.

0.4.1. It can happen that all subwords of a superword W are permitted, but $W \notin S$. An example of such a word is already given in 0.1: it is superword $W = \dots vvhvv \dots$ of letters v with the only h.

0.4.2. Superwords W_L and W_{L^*} may coincide even if lines L and L^* do not: this will be the case if L^* is symmetric to L (taking orientation into account) with respect to a vertical, a horizontal or a knot of the grid; it will also be the case if the lines are parallel and L^* is L shifted by 1 (or any integer) along vertical or horizontal direction.

If we want that different words correspond *generically* (see 0.4.3) to different lines L and L^*, we can fix a square K_0 and consider only lines with positive slopes that pass through K_0 and cross the same side of it.

0.4.3. Let lines L and L^* have the same slope k_L and let none of them pass through grid knots. If, moreover, k_L is rational, then (as it is readily seen) superwords W_L and W_{L^*} coincide.

If instead k_L is irrational, then the set of all subwords in W_L and W_{L^*} coincide, while W_L and W_{L^*} need not coincide (and they do not if L and L^* pass through the same side of square K_0, see 0.4.2).

At this stage – solving preparatory problems – we will only formulate the above statement; try to prove it on your own. Also try to prove another fact – *quasiperiodicity* of W_L: if slope k_L is irrational then W_L contains arbitrary long coinciding subwords.

0.5.1. Erase one v in front of each h in $W \in S$ from 0.3. The resulting superword $\{\dots hvvhvh \dots\}$ remains in class S. In general case this fact is proved in 0.5.2. Intuition for 0.5.2: if lines L_1 and L_2 have slopes $k_{L_1} = 3/4$ and $k_{L_2} = 3/5$, then W_{L_1} and W_{L_2} are periodic superwords with periods $\{hvvhvhv\}$ and $\{hvvhvvhv\}$ (compare to 0.3.4).

0.5.2. Let line L with slope $k_L < 1$ cross some horizontal g at point P. Erase one v in front of each h in superword W_L. the resulting superword W^* remains in class S since $W^* = W_{L^*}$ for line L^* that passes through P with slope k_{L^*} such that $1/k_{L^*} = 1/k_L - 1$.

0.6. Let line L with slope $k_L < 1$ cross some horizontal g at point P. Insert one v in front of each h in superword W_L. Then the resulting superword W^* remains in class S since $W^* = W_{L^*}$ for line L^* that passes through P with slope k_{L^*} such that $1/k_{L^*} = 1/k_L + 1$.

0.7. If line L has slope k_L then

$W_L \in E$ if and only if $k_L = 1$;

$W_L \in S^h_\infty$ if and only if $k_L = 0$;

$W_L \in S^v_\infty$ if and only if $k_L = \infty$;

$W_L \in S^h_n$ if and only if $k_L \in (1/(n+1), 1/n]$; in this case $k_L = 1/n$ if and only if there are n letters v between any two consecutive letters h.

$W_L \in S^v_n$ if and only if $k_L \in [n, n+1)$; in this case $k_L = n$ id and only if there are n letters h between any two consecutive letters v.

0.7.1. Geometric interpretation of 0.7 in terms of strip polygons is as follows: lengths λ_s of strip polygons M_w for a permitted word w that is a subword of a superword $W \in S^h$ can take at most two values:

all lengths $\lambda_s = n+1$ if there are n letters v between any two consecutive letters h;

lengths λ_s are equal either to $n+1$ or to $n+2$ if $W_L \in S^h_n$ and $k_L \neq 1/n$.

if strip lengths λ_s of M_w take more than two values, then word w is *necessarily* forbidden. if strip lengths M_w take only two values, word w may *still* be forbidden (see 0.7.2). For superwords $W \in S^v$ one must substitute $v \rightleftarrows h$ before counting the number of λ_s values.

0.7.2. Here is an example of a periodic forbidden word w for which lengths λ_s of strips in M_w take two values 2 and 3. It has the following ten-letter period: $\{hvhvhvvhvv\}$.

Check that corresponding polygon M_w can not be crossed by a single line.

0.8.1. Let the slope k_L of line L be equal to a *finite* continued fraction, for example, one of the following:

$$\frac{11}{25} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}}}, \quad \frac{7}{24} = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3}}}, \quad \frac{4}{19} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{3}}}, \quad \frac{5}{8} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}.$$

For each of these fractions let us find the chain $a_1, a_2, \dots, a_n$ for superword $W = W_L$.

If $k_L = 11/25 \in (1/3, 1/2]$ then $W = W_L \in S_2^h$ (see 0.7).

Hence $a_1 = 2$. Erase a_1 letters v in front of each h and denote the resulting superword by W_1. By 0.7 and 0.5.2 we have:

$W_1 = W_{L_1} \in S_3^v$ since $25/11 - 2 = 3/11 \in (1/4, 1/3]$.

Hence $a_2 = 3$. Erase a_2 letters h in front of each v. We get $W_2 = W_{L_2} \in S_1^h$ since $11/3 - 3 = 2/3 \in (1/2, 1]$.

Hence $a_3 = 1$. Erase a_3 letters v in front of each h and denote the resulting superword by W_3. Now $W_3 \in S_2^v$ because $3/2 - 1 = 1/2$.

Hence $a_4 = 2$. Erase a_4 letters h in front of each v. We get $W_4 \in S_\infty^h$. Therefore, if $k_L = 11/25$ then $a_1 = 2$, $a_2 = 3$, $a_3 = 1$, $a_4 = 2$.

Corollary. If $k_L = 11/25$ then W_j are the superwords with the following periods:

$W_4 = \{\dots v \dots\}$; $W_3 = \{\dots hhv \dots\}$; $W_2 = \{\dots vvhvh \dots\}$; $W_1 = \{\dots vh^4vh^4vh^3 \dots\}$;

$W = \{\dots hv^2hv^2hv^2hv^3\ hv^2hv^2hv^2hv^3\ hv^2hv^2hv^3 \dots\}$.

Similarly, for $k_L = 7/24$ we have $a_1 = 3$, $a_2 = 2$, $a_3 = 3$; if $k_L = 4/19$ then $a_1 = 4$, $a_2 = 1$, $a_3 = 3$, and if $k_L = 5/8$ then $a_1 = a_2 = a_3 = 1, a_4 = 2$.

0.8.2. If slope k_L of line L is rational:

$$k_L = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n}}}},$$

then reorganizations of the superword $W = W_L$, corresponding to line L, stop after finitely many steps: $W \to W_1 \to W_2 \to \dots \to W_n$ and result in series $a_1, a_2, \dots, a_n$.

1.1. The simplest examples of carefully mixed words are

$$\{\dots aaa \dots\} \text{ and } \{\dots bbb \dots\}.$$

A second simplest example: superword $W = W_L \in S$ for line L with slope k_L equal to either n or $1/n$. For instance, let $k_L = 1/n$. By 0.7 in $W = W_L$ in this (and only this) case there are n letters v between any two consecutive h:

$$W = \{\dots hvhvhv \dots\} \text{ for } n = 1,\ W = \{\dots hvvhvvhvv \dots\} \text{ for } n = 2, \text{ etc.}$$

That such W are carefully mixed is obvious: $W = \{\dots hv^nhv^nhv^n \dots\}$ any subword of length $r \le n$ contains either r letters v, or one letter h and $r - 1$ letters v; any subword of length $(n+1)k + r$, where $0 \le r \le n$, contains either k letters h and $nk + r$ letters v, or $k + 1$ letters h and $nk + r - 1$ letters v.

1.2.1. Any permitted word w is, obviously, a subword of infinitely many different superwords W_L from S: if polygon M_w can be crossed by a line L that does not pass through knots then it can as well be crossed by any line L^* close enough to L.

1.2.2. In particular, line L^* from 1.2.1 can be chosen with rational slope k_L.

1.2.3. It follows from the definition of a carefully mixed word that a superword W is carefully mixed if and only if each its subword is carefully mixed.

1.3.1. Answer: *no.* Replace letters a and b with h and v. Here is a counterexample:

$w = hvvhvvhvhv$ is permitted while $w^* = vvhh$ is forbidden.

To elaborate a little: $w = hvvhvvhvhv$ is permitted since $w' = hvvhvvhvhvv$ is permitted (and $w' = hvvhvvhvhvv$ is permitted since $w'^* = vvhv$ is permitted, see the theorem below).

The following theorem holds: *renaming maps*

1) superwords $W \in S$ to superwords $W^* \in S$,

2) forbidden words W to forbidden words W^*,

3) superwords $W \notin S$ to superwords $W^* \notin S$.

We sketch the proof.

Let M_W be the strip polygon for word W. Boundaries of M_W are broken lines $\underline{L}$ and $\overline{L}$, with units along verticals and horizontals of the grid. For subwords of W of types hv^n and hv^{n+1} adjacent units of these broken lines are legs of right triangles of two kinds. Hypotenuses of these triangles form new broken lines $\underline{L}'$ and $\overline{L}'$ inside M_W. Units of both these new broken lines lie on a *new* grid generated by lines with slopes $1/n$ and $1/(n+1)$ (the knots of the new grid are the same as of the old one), so each unit corresponds either to word hv^n or to word hv^{n+1}.

Introduce coordinates x, y on the grid and consider transformation F that maps point $\{x, y\}$ into $\{x-ny, -x+(n+1)y\}$. This transformation maps vectors $\{n+1, 1\}$ and $\{n, 1\}$ to vectors $\{1, 0\}$ and $\{0, 1\}$, broken lines $\underline{L}'$ and $\overline{L}'$ to broken lines $\underline{L}^*$ and $\overline{L}^*$ with units along horizontals and verticals of the grid.

Therefore transformation F corresponds to renaming (replacing words hv^{n+1} with letters v and words hv^n with letters h): if we denote by M^* the polygon bounded by broken lines $\underline{L}^*$ and $\overline{L}^*$ then it will turn out to be the strip polygon for word W^*.

In what follows we need the following properties of linear transformations F:

F maps lines to lines and preserves ratios of segment lengths.

1.3.2. Solutions of problem 1.3.2 *are still accepted*.

We will need the following theorem about superwords:

Renaming maps superwords $W \in S_1$ to superwords $W^* \in S_1$ and superwords $W \notin S_1$ to superwords $W^* \notin S_1$.

Let us prove the first part of the theorem.

By hypothesis superword W can be split into subwords of two types: ab^n and ab^{n+1} (when you solve problem 1.3.2 you may conveniently relate to them as *short and long segments*). Now perform the following *renaming*: we replace each subword ab^{n+1} by b and each subword ab^n by a. Denote by W^* the resulting image of W.

Introduce the following notation for each word w:

$a(w)$ is the number of letters a in w, $b(w)$ is the number of letters b in w; $|w|$ is the length of w.

Remark. Below we draw on the following statement (prove it):

In any word (or superword) $W^* \notin S_1$ there exist two subwords U^* and V^* of the same length for which $b(U^*) - b(V^*) = 2$.

Consider a superword $W \in S_1$. We want to prove that superword $W^* \in S_1$. Assume the converse: suppose there exist two subwords U^* and V^* in W^* such that $|U^*| = |V^*|$ and $b(U^*) - b(V^*) = 2$. The last two equalities imply that the following holds for words U and V that are mapped to U^* and V^* by renaming transformation:

$$|U| - |V| = b(U) - b(V) = 2.$$

Of subwords U and V build two subwords U' and V' of superword W.

Words U and V consist of segments ab^n and ab^{n+1}; word U' is U with erased first a and word V' is V with another a at the tail. Hence

$$|U'| = |V'|, \quad b(U') - b(V') = 2.$$

That such U' and V' are subwords of W contradicts the hypothesis that $W \in S_1$. Therefore $W^* \in S_1$ is proved by contradiction.

1.4.1. Let us prove that each permitted word W is carefully mixed. Let U and V be any two its subwords of the same length such that U has n letters h and m letters v while V has $n+d$ letters h and $m-d$ letters v where $d \geq 0$. Assume that the strip polygon of W can be crossed by a line L with slope k. Since this line crosses the polygon of word U, we have $\frac{m-1}{n+1} < k$. On the other hand, crossing the polygon of V gives $k < \frac{m-d+1}{n+d-1}$. Therefore $\frac{m-d+1}{n+d-1} > \frac{m-1}{n+1}$, or $(n+m)(2-d) > 0$. Hence $d < 2$, i.e., W is carefully mixed.

1.4.2. Any superword $W \in S$ is carefully mixed since, by 1.4.1, every its subword is carefully mixed and by 1.2.3 a superword is carefully mixed if and only if so are all of its subwords.

1.4.3. We have already come across an example of carefully mixed superword $W \notin S$ twice - see solutions of 0.1 and 0.4.1: it is superword $W = \dots vvhvv \dots$ of letters v with the only letter h.

1.5. lets prove that for any relatively prime m and n there exists a periodic carefully mixed superword of letters a and b with m letters a and n letters b in its period.

Consider $W_L \in S$, where L is a line with slope m/n. To get W out of W_L it suffices to substitute $h \to a$, $v \to b$ in W_L.

If m and n are not relatively prime then there exists no periodic carefully mixed superword W such that its *shortest* period has m letters a and n letters b. Take, for example, $m = 2$, $n = 4$. Then for

$$W = \{a\,b\,b\,a\,b\,b\} = \{b\,a\,b\,b\,a\,b\} = \{b\,b\,a\,b\,b\,a\}$$

the period of length six is not the shortest and none of the other superwords with periods of two letters a and four letters b is carefully mixed since they all have subwords aab and bbb.

If we relax the requirement that the period be the shortest then the statement is true for all m and n, not necessarily relatively prime.

1.6.1. Solutions of 1.6.1 *are still accepted.*

1.6.2. Solutions of 1.6.2 *are still accepted.*

1.7. We start describing superwords W that are in S_1 but not in S. It will bring us for problems 3.1.1-3.1.3 of the main list; solving them will help you continue the description.

According to 1.2.3, a superword $W \in S_1$ if and only if every its subword w is carefully mixed. By 1.6.2 any carefully mixed word w is permitted. Hence for each $w \in W$ strip polygon M_w can be crossed by a line $L(w)$ that does not pass through knots.

Consider an embedded sequence $w_1 \subset w_2 \subset \dots$ of subwords $w_n \in W$, that converges to W (meaning that each letter of W belongs to all w_n for n large enough). Corresponding to it is an embedded sequence of strip polygons M_n with common square K_0. Each of these polygons can be crossed by a line L_n passing through K_0. Let $y = k_n x + b_n$ be the equation of L_n. Sequences k_n and b_n are bounded (check it). Hence we can choose a subsequence of pairs k_{n_j}, b_{n_j} converging to k, b.

Line L given by equation $y = kx + b$ has a common point with each of the strip polygons M_n. It must pass through at least one knot of the grid (why?).

To be continued in problems 3.1.1-3.1.3.

1.8. Generalize the definition of carefully mixed superwords of alphabets of more than two letters: $a, b, c, \ldots$. Superword W is carefully mixed if the number of letters a in any two of its subwords U and V of the same length differs at most by one1; we require the same for letters b, letters c and so on.

Here is an example of a *periodic* carefully mixed superword of letters a, b, c containing the superword aa: $W = \{\ldots\{a\,a\,b\,a\,c\,a\,b\,a\,c\}\ldots\}$. To see that W is carefully mixed we construct three auxiliary superwords of letters v, h.

The first is W where a is replaced by v, b and c by h: $W_1 = \{\ldots\{v\,v\,h\,v\,h\,v\,h\,v\,h\}\ldots\}$.

The second is W where a and c are replaced by v, b by h: $W_2 = \{\ldots\{v\,v\,h\,v\,v\,v\,h\,v\,v\}\ldots\}$.

The third is W where a and b are replaced by v, c by h: $W_3 = \{\ldots\{v\,v\,v\,v\,h\,v\,v\,v\,h\}\ldots\}$.

All three auxiliary words belong to S:

$$W_1 = W_{L_1}, \quad W_2 = W_{L_2}, \quad W_3 = W_{L_3}$$

for lines L_1, L_2, L_3 with slopes $k_{L_1} = 4/5$, $k_{L_2} = k_{L_3} = 2/7$.

Hence all of them are carefully mixed; in particular, the number of letters v in any two subwords U and V of W_1 of the same length differs at most by one. But then the number of letters a in any two subwords U and V of W of same length differs at most by one. Similar argument (applied to subwords of W_2 and W_3, respectively) shows that so do the numbers of letters b and letters c in any two subwords of W which completes that proof.

2.1. By $T_W(n)$ denote the number of different subwords of length n in superword W.

Maximum possible value of $T_W(n)$ for a two-letter alphabet is obviously equal to 2^n (3^n for a three-letter alphabet, m^n for an m-letter alphabet).

2.2. Suppose $T_W(k) = T_W(k+1) = T$ for some k (that is, the number of different subwords of length k in superword W is equal to the number of different subwords of length $k+1$ denoted by T).

Let us prove that in this case superword W is periodic with period T.

For each d every subword of length d in superword W can be extended to the right to a subword of length $d+1$ at most in two ways (with either a or b). Equality $T_W(k) = T_W(k+1)$ means that for $d = k$ only one way exists: if a subword of length k could be continued in two different ways we would end up with more than $T_W(k)$ words of length $k+1$. Likewise any two coinciding subwords of W of length k have the same letter to the left of them.

Write down $T + k$ consecutive letters $w_1, \ldots, w_{T+k}$ of superword W and (respectively) $T+1$ subwords $W_0, \ldots, W_T$ of length k composed of these letters.

$$W_0 = \{w_1, \ldots, w_k\}, \; \ldots, \; W_T = \{w_{T+1}, \ldots, w_{T+k}\}.$$

Since the number of different subwords of length k in superword W is T, we have $W_i = W_j$ for some i and j:

$$W_i = \{w_{i+1}, \dots, w_{i+k}\} = W_j = \{w_{j+1}, \dots, w_{j+k}\},\ 0 \le i < j \le T,$$

or, equivalently,

$$W_i = \{w_{i+1}, \dots, w_{i+k}\} = W_{i+P} = \{w_{i+P+1}, \dots, w_{i+P+k}\},\ 1 \le P = j - i \le T.$$

Since any subword of length k of superword W can be uniquely expanded to the right and to the left, the above implies that

$$W_s = \{w_{s+1}, \dots, w_{s+k}\} = W_{s+P} = \{w_{s+P+1}, \dots, w_{s+P+k}\},\ \text{ for any integer } s,$$

i.e. superword W is periodic and the length of period $P \le T$. Since the number of k-letter subwords of such a superword is P and by hypothesis it is T, we have $P = T$.

Remark. Therefore, $i = 0, j = T$ and

$$W_0 = \{w_1, \dots, w_k\} = W_T = \{w_{T+1}, \dots, w_{T+k}\}.$$

2.3. Suppose $T_W(k) < k + 1$ for a certain k. Let us prove that then W is periodic.

If $W = \{\dots aaa \dots\}$ or $W = \{\dots bbb \dots\}$ (if $T_W(k) = 1 \le k$ for all k) then the statement holds: W is periodic.

Let us now suppose that W contains both letters a and letters b. Then

$$2 \le T_W(1) \le T_W(2) \le \dots \le T_W(k) \le k,$$

Since k numbers $T_W(i)$ take $k - 1$ values, there exits i such that $T_W(i + 1) = T_W(i)$. The periodicity of W follows now from 2.2.

2.4. Let us prove that for all $W \in S$ the following inequality holds:

$$T_W(k) \le k + 1 \text{ for all } k.$$

2.4.1. We start with description of $T_W(k)$ for $W = W_L$ in case of rational slope $k_L = m/n$ (for mutually prime m and n).

In this case, as it can be easily seen, $T_W(k) = k + 1$ for $k < P = m + n$ and $T_W(k) = P$ for $k \ge P$, so for rational k_L inequality $T_W(k) \le k + 1$ is proved for all k.

2.4.2. Suppose now that k_L is irrational. Let us prove that $W = W_L$ satisfies $T_W(k) = k + 1$ for all k and thus $W \in S_2$.

By 2.3 we have $T_W(k) \ge k + 1$ for all k. What is left is to prove that for no k inequality $T_W(k) > k + 1$ can hold. Assume the converse and take one k for which it holds. Let w be a subword of superword W having $k + 2$ subwords of W of length k.

By 1.2.2 there exists a line L^* with *rational* slope that crosses strip polygon M_w. Word w is a subword of superword $W^* = W_{L^*}$, i.e., $T_{W^*}(k) \ge k + 2 > k + 1$ against 2.4.1. This contradiction implies that our assumption was wrong and hence $W = W_L$ satisfies $T_W(k) = k + 1$ for all k, i.e., $W \in S_2$.

2.4.3. An example of a superword $W \in S_2$ not belonging S already showed up in solutions for 0.1: $\dots vvvhhh \dots$ (first letters v only, then letters h only).

2.4.4. All superwords W such that $W \in S_1$ and $W \notin S$, belong to S_2.

To be continued in problems 3.3, 3.4.1-3.4.3 of the main list.