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LINES ON GRID PAPER AND SUPERWORDS

Main Problems

Completion of class S. Class S_0: Sturm sequences. When we were defining class S, we were drawing lines on grid paper not passing through knots. Let us now draw a line L with *irrational* slope k_L *through a knot* . As usually, we write letters v and h on each intersection of L with verticals and horizontals and put a question mark '?' on the knot. Substituting vh for '?' gives superword $W = W_L$; substituting hv for '?' gives superword $\overline{W} = \overline{W}_L$.

3.1.1. Prove that the resulting superwords are carefully mixed: $W, \overline{W} \in S_1$.

3.1.2. Prove that $W, \overline{W} \notin S$.

Add to S all resulting superwords W_L, $\overline{W}_L$ (for all irrational slopes k_L) and denote by S_0 so completed class of superwords. Superwords $W \in S_0$ are called *Sturm sequences* or *Sturm superwords* after French mathematician Jacques Sturm (1803–1855).

3.1.3. Find all carefully mixed superwords not in S_0.

Hint: make use of the following construction. Draw a line L with *rational* slope k_L *through a knot*. Write v's and h's on all intersections of L with verticals and horizontals and question marks on all knots. Substitute vh's for some question marks and hv's for the others. Find out which substitutions result in (a) superwords $W \in S$, (b) superwords $W \in S_1$, (c) superwords $W \notin S_1$.

Exceptional superwords. Superword magnification. Consider three superwords: $W_0 = \{\dots vvv \dots\}$, $W_1 = \{\dots vvvhvvv \dots\}$, $W_2 = \{\dots vvvhhh \dots\}$.

3.2.1. Check that

$W_0 \in S_0 \subset S_1, W_0 \notin S_2;\quad W_1 \in S_1, W_1 \in S_2, W_1 \notin S_0;\quad W_2 \in S_2, W_2 \notin S_0, W_2 \notin S_1$

(that is, W_0 is a carefully mixed Sturm sequence but is not a superword with slowly growing subword sets; W_1 is a carefully mixed superword with slowly growing subword sets but is not a Sturm sequence; W_2 is a superword with slowly growing subword sets but is neither carefully mixed nor a Sturm sequence).

3.2.2. Is it true that there exists the only superword in S_2 (namely, superword W_2) that contains subword $vvhh$?

Superwords W which (like W_0, W_1 and W_2) have one of the following three properties:

$W \in S_0 \subset S_1, W \notin S_2,\quad W \in S_1, W \in S_2, W \notin S_0,\quad W \in S_2, W \notin S_0, W \notin S_1,$

are called *exceptional*. Consider two operations with superwords: *regular n-substitution*: $hv^n \to v, hv^{n+1} \to h$ (each v is replaced by hv^n, each h – by hv^{n+1}, where n is any nonnegative integer) and *inversion*: $h \rightleftarrows v$.

Successive application of (finitely many of) these operations to a word or a superword is called *magnification*; for different n-substitutions n need not be the same.

3.3. Prove that any exceptional superword is magnification of one of our W_j, $j = 0, 1, 2$.

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Farey series[1], basic parallelograms and exceptional $W \in S_2$.

Farey series F_j are defined as the sets of all fractions on $[0,1]$ in lowest terms whose denominators do not exceed j, arranged in order of magnitude:

$$F_1 = \{0,1\}, \quad F_2 = \{0, \frac{1}{2}, 1\}, \quad F_3 = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\},$$

$$F_4 = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\}, \quad F_5 = \{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\}, \quad \dots$$

Expressing 0 and 1 as $\frac{0}{1}$ and $\frac{1}{1}$ guarantees that for any two adjacent fractions $\frac{x_1}{y_1} < \frac{x_2}{y_2}$ of any F_j the condition $x_2y_1 - x_1y_2 = 1$ holds. It means (check it!) that the parallelogram Π spanned by vectors $\{x_1, y_1\}$ $\{x_2, y_2\}$ has unit area.

Such parallelograms Π with vertices at knots are called *basic*.

3.4.1. Relate to each pair of rational numbers k_1, k_2 such that

$$0 \le k_1 = \frac{x_1}{y_1} < k_2 = \frac{x_2}{y_2} \le 1,$$

a) parallelogram Π, spanned by vectors $\{x_1, y_1\}$ and $\{x_2, y_2\}$,

b) two periodic superwords $W_1 = W_{L_1}, W_2 = W_{L_2} \in S$ for lines with slopes k_1, k_2 and hence a superword W composed (as in 2.7) from two periodic parts W_1, W_2.

Prove that $W \in S_2$ if and only if parallelogram Π is basic (or, equivalently, when k_1, k_2 are adjacent fractions in a certain F_j).

3.4.2. Explain why all $W \in S_2$, related to basic parallelograms in 3.4.1, are exceptional: they are not in S_1.

3.4.3. Prove that each exceptional superword $W \in S_2$ is related to a basic parallelogram Π: two rays parallel to the sides of Π, cross strip polygon M_w for each subword $w \in W$ (perhaps after applying to w inversion $v \rightleftarrows h$).

Canonical representation of periodic superwords W_L. We know that, for lines with rational slopes k_L, superwords $W_L \in S_0$ are periodic. If $k_L = m/n$, where m/n is in lowest terms, then period P of superword W_L has m letters h and n letters v, but arranging these letters in right order is not an easy task for large m and n. Farey series help find the following *canonical* representation of period P as a word $f_{m,n}$ of m letters h and n letters v (we calls such words $(m,n)-$*fareys*): by definition $(0,1)-$farey $f_{0,1}$ is one-letter word v, $(1,1)-$farey $f_{1,1}$ is two-letter word vh. If $(m,n)-$fareys are already defined for all $m \le n < q$, then $(m,q)-$fareys are derived from $(i,j)-$fareys, $j < q$, by means of the following operation called *concatenation*: each fraction $\frac{m}{q}$ in series F_q has two adjacent fractions $\frac{x_1}{y_1}$ and $\frac{x_2}{y_2}$ for which $\frac{m}{q}$ is *the mediant*: $x_1 + x_2 = m$, $y_1 + y_2 = q$. Denominators of these two fractions are smaller than q, so f_{x_1,y_1} and f_{x_2,y_2} are already defined. Then $f_{m,j}$ is derived by writing f_{x_2,y_2} next to f_{x_1,y_1}. Here are some examples: $f_{1,2} = f_{0,1} \cup f_{1,1} = vvh$, $f_{1,3} = f_{0,1} \cup f_{1,2} = vvvh$, $f_{2,3} = f_{1,2} \cup f_{1,1} = vvhvh$, $f_{1,4} = f_{0,1} \cup f_{0,3} = vvvvh$, $f_{3,4} = f_{2,3} \cup f_{1,1} = vvhvhvh$.

3.5.1. Check that

$f_{1,5} = vvvvvh$, $f_{2,5} = vvvhvvh$, $f_{3,5} = vvhvvhvh$, $f_{4,5} = vvhvhvhvh$.

[1]These series were invented in 1816 by British geologist John Farey (1766-1826).

3.5.2. Prove that $(m, n)-$fareys are well defined: when $k_L = m/n$ superwords $W = W_L$ indeed have period $P = f_{m,n}$.

3.5.3. Let $\frac{x_1}{y_1} < \frac{x_2}{y_2}$ be two adjacent fractions for a certain series F_j and $y_1 < y_2$. Prove that in this case f_{x_1,y_1} is a subword of f_{x_2,y_2} (likewise, if $y_1 > y_2$ then f_{x_2,y_2} is a subword of f_{x_1,y_1}).

Golden ratio. *Golden ratio* is the split of a segment into two such that the ratio of the length of the *entire* segment to the length of the *bigger* part is equal to that for the *bigger* to the *smaller* part: $(\frac{\sqrt{5}+1}{2} : 1 = 1 : \frac{\sqrt{5}-1}{2})$.

3.6.1. Draw ray ℓ with slope $\frac{\sqrt{5}-1}{2}$ through the origin.

Write v's and h's on all intersections of ℓ with verticals and horizontals; we get *half of a superword* $w_\ell = \{v\,v\,h\,v\,v\,h\,v\,h\,\ldots\}$ (i.e., sequence w_ℓ infinite to the right). Prove that w_ℓ can be constructed in the following way:

In three-letter word $v\,v\,h$ substitute

$$v\,v\,h \to v, \quad v\,h \to h.$$

(each v is replaced by $v\,v\,h$, each h — by $v\,h$). repeat the same substitution for the resulting word $v\,v\,h\,v\,v\,h\,v\,h$: $v\,v\,h\,v\,v\,h\,v\,h\;v\,v\,h\,v\,v\,h\,v\,h\;v\,v\,h\,v\,h$ and so on.

Remark: one may start even from the single letter v and still get w_ℓ after *infinitely many* (compare to 3.3) substitutions.

3.6.2. Formulate and solve similar problem for ray ℓ with slope $k_\ell = \sqrt{2} - 1$.

3.6.3. Expand the result onto slopes k_ℓ equal to periodic continued fractions $[a_1, \ldots, a_n \ldots]$, $[[b_1, \ldots, b_n \ldots]]$ with periods $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$.

4. Sturm sequences. Split circle C of unit length into two arches, A and B, with lengths α and β, $\alpha+\beta = 1$. Choose an initial point P_0 and construct sequence $P_0, P_1, P_2, \ldots$, where each point P_k is P_{k-1} shifted by β along circle C, and sequence $P_0, P_{-1}, P_{-2}, \ldots$, where each point P_{-k} is $P_{-(k-1)}$ shifted by $-\beta$ along circle C.

In other words, P_n is P_0 shifted by $n\beta$ along C, $n = \pm1, \pm2, \ldots, \pm k, \ldots$.

To points P_n we relate sequence $W = \{w_n\}$ of letters a and b, where

$$w_n = a, \text{ if } P_n \in A, \text{ and } w_n = b, \text{ if } P_n \in B, \;\; n = 0, \pm1, \pm2, \ldots.$$

4.1. Prove that the class of such sequences $W = \{w_n\}$ coincides (after replacing a and b by v and h) with class S_0 of Sturm sequences $W = W_L$.

Hint 1: You should view each of the arches A and B as *semiopen* — containing one of the endpoints but not the other.

Hint 2: The following three (simpler) problems will help you with 4.1:

4.1.1. Which of subwords aa, ab, ba, bb of length 2 can be contained in W?

4.1.2. Can W contain three-letter subword abb?

4.1.3. Let $0 \le s, t \le 1$, $w_n = [sn + t] - [s(n-1) + t]$, where square brackets represent the integer part of a number.

Check that w_n can have only two values 0 and 1.

Is it true that for all s and t on $[0, 1]$ sequence $W = \{w_n\}$ is the one corresponding to points P_n (after substituting a for 0 and b for 1), and a subword from S_0 (with 0 replaced by v and 1 by h)?

4.2. Describe Sturm sequences $W \in S_0$ not in S in terms of α, β and P_0.

4.3. Prove that if β is irrational then the set of points P_n is *dense everywhere* and *evenly distributed* on circle C. This means the following.

On circle C take an arch d of length δ, and find how many of any N consecutive points from $\{P_n\}$ hit arch d. Then the number of hits, divided by N, tends to δ as $N \to \infty$.

Superword equivalence. Two superwords are called *equivalent*, if they have exactly the same subword sets.

4.4. Prove that for fixed α and β, $\alpha+\beta = 1$ and varying P_0 Sturm sequences $W = \{w_n\}$ are equivalent.

4.5. Prove that for irrational α and β, $\alpha + \beta = 1$, Sturm sequences $W = \{w_n\}$ are *quasiperiodic*: if a subword occurs in such a sequence, it occurs there infinitely many times.

4.6. Prove that for fixed α and β, $\alpha + \beta = 1$ and $P_0^* = P_0 + k\beta$ (for integer k) Sturm sequences W, W^* are shifts of each other.

4.7. Proved that for fixed irrational α and β, $\alpha+\beta = 1$ and $P_0^* \neq P_0+k\beta$ (for integer k) Sturm sequences W, W^* are neither congruent nor shifts of each other.

5. Selfdeveloping superwords.

5.1. Consider two sequences of words U_k and V_k of letters a and b:

$U_0 = a,\ V_0 = b$ и $U_k = U_{k-1}^2 V_{k-1},\ V_k = U_{k-1} V_{k-1}$ for all $k \geq 1$;

the last formula means that to get U_k one writes U_{k-1} twice and then V_{k-1}; to get V_k one writes U_{k-1} and then V_{k-1}. Hence U_k begins with U_{k-1} and V_k ends with V_{k-1} for all $k \geq 1$:

$$U_0 = a,\ U_1 = aab,\ U_2 = aab\ aab\ ab,\ \dots \quad V_0 = b,\ V_1 = ab,\ V_2 = aab\ ab,\ \dots$$

and therefore words $W_k = V_k U_k$ form an embedded sequence and *selfdevelop* to the superword W which is the union of them all:

$$W_0 = b\,a \subset W_1 = ab\,aab \subset W_2 = aab\,ab\ aab\,aab\,ab \subset \dots \subset W = \dots aab\,ab\ aab\,aab\,ab \dots .$$

Check that the resulting superword W can be split into subwords aab and ab:

$$W = \dots aab\,aab\,ab\,aab\,ab\ aab\,aab\,ab\,aab\,aab\,ab\,aab\,ab \dots .$$

5.1.1. Prove that superword W is not periodic.

5.1.2. In W replace each aab by v and then each ab by h: $W = \dots v\,v\,h\,v\,h\ v\,v\,h\,v\,v\,h\,v\,h \dots$

Is it true that W is preserved if we substitute $vvh \to v$, and then $vh \to h$?

5.1.3. is it true that superword W is equivalent to a superword from class S, i.e., there exists a line L such that $W \sim W_L$? If so, what is its slope k_L?

5.2. Pick any sequence $\{n_k\}$ of positive integers and relate to is two sequences of words U_k and V_k in which

$U_0 = a,\ V_0 = b,\ U_1 = U_0^{n_1+1} V_0,\ V_1 = U_0^{n_1} V_0,\ U_2 = U_1^{n_2+1} V_1,\ V_2 = U_1^{n_2} V_1,$ and, in general,

$$U_k = U_{k-1}^{n_k+1}\, V_{k-1},\ V_k = U_{k-1}^{n_k}\, V_{k-1} \text{ for all } k \geq 1.$$

Prove that (as in 5.1) words $W_k = V_k U_k$, developing "from the middle" to the left and to the right, form a both ways infinite selfdeveloping superword W.

5.2.1. In 5.1, which is a special case of 5.2, all $n_k \equiv 1$.

Consider another special case: check that for $n_k \equiv 2$ resulting superword W can be split into words $a^3 b = aaab$ and $a^2 b = aab$:

$$W = \dots aaab\,aaab\,aab\ aaab\,aaab\,aaab\,aab \dots .$$

Prove that this W is not periodic.

In W replace each $aaab$ by v and then each aab by h:

$$W = \dots v\,v\,h \;\; v\,v\,v\,h\,\dots$$

is it true that W is preserved if we substitute $vvvh \to v$ and then $vvh \to h$?

Is it true that the resulting W is equivalent to a superword from class S, i.e., there exists a line L, such that $W \sim W_L$?

If so, what is the slope k_L of L and what is its equation?

5.2.2. Is it true that for any sequence $\{n_k\}$

a) W is carefully mixed,

b) $T_W(n) = n + 1$ for all n,

c) W is a Sturm sequence?

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