A. Belov-Kanel, M. Gerver

Lines on Grid Paper and Superwords

Let line L on the grid paper not pass through its knots. let us write letters v and h on all intersections of L with verticals and horizontals of the grid, respectively.

Now we have *words* composed of letters v and h on all finite segments of L; on the entire line L there appears a sequence W_L which is infinite both ways.

Such sequences W_L are called *superwords.*

We consider line L *oriented*: a direction on L is chosen. If we switch it then superword W_L will read in the opposite direction. For example, $\{\dots vvh\,vvh\,vh\dots\}$ will look like $\{\dots hv\,hvv\,hvv\dots\}$.

Main questions: What singles W_L out of all two-letter superwords? And how combinatorial properties of W_L are related to geometric properties of line L and arithmetical properties of slope and intercept of L?

Announcement: Shortly we will present a procedure that relates superword W_L with the continued fraction expansion of the slope of line L.

The following examples give an idea of what a continued fraction is:

$$\frac{11}{25}=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\frac{1}{2}}}},\quad \frac{5}{8}=\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}},\quad \frac{\sqrt{5}-1}{2}=\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}},$$

$$\frac{8}{5}=1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}},\quad \frac{\sqrt{5}+1}{2}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}},\quad \sqrt{2}+1=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}.$$

A little later we will talk about continued fractions and their geometric interpretation (*nose stretching* algorithm) in more detail. Now concentrate on the following question: if the slope of L equals one of the above *periodic* continued fraction, how can this periodicity be seen in W_L?

Terminology. The following terminology helps us to be concise and precise. A *word* is a finite sequence of characters; a *superword* is an infinite (both ways) sequence of characters. A *subword* is a word that is a fragment of another word (or superword). The *length* of a word is the number of characters in it. For example, words vv, vh, aba, 001 are some of the subwords of length 2 and 3 in superwords $\{\dots vvhvvhv\dots\}$, $\{\dots aabaaba\dots\}$, $\{\dots 0010010\dots\}$.

Class S. By S denote the class of all superwords $W=W_L$ corresponding to all oriented lines L that pass through no knots on the grid paper.

Alongside with S we consider a few more classes of superwords: carefully mixed superwords, superwords with slowly growing subword sets, and others.

Comparing these classes with S we will find that W_L exhibit a number of beautiful properties.

Further questions await an interested reader, among them there are unsolved problems.

Preparatory problems

Some of the following problems are really simple, others are more difficult.

Hint: if you get stuck with a problem, move on and come back when you have more problems solved.

0.1. Give an example of word W of letters v, h, that certainly does not belong to S.

0.2.1. For which lines L superwords W_L from class S are periodic?

0.2.2. For which lines L letters v come in superwords $W_L \in S$ more often than letters h?

0.2.3. Substitute $h \rightleftarrows v$ in a superword W_L from S. Will the resulting superword W' remain in S? In other words, is it always true that $W' = W_{L'}$ for some line L'? If so, how is line L' related to L?

Strip polygons M_w. Fix a square K_0 on grid paper and to each word w of n letters w_j

$$w = \{w_1, \ldots, w_n\} \text{ (each letter } w_j \text{ is either } v \text{ or } h),$$

relate a polygon M_w that consists of $n+1$ squares $K_0, K_1, \ldots, K_n$, according to the following rule. Square K_1 is adjacent to K_0 from above if $w_1 = h$ and from right if $w_1 = v$. Similarly, if $j-$th letter is $w_j = h$ then K_j is adjacent to K_{j-1} from above (their common side is *horizontal*) and if $w_j = v$ then K_j is adjacent to K_{j-1} from right (their common side is *vertical*).

We will call such polygons M_w *strip polygons.* Polygon M_w lies in several consecutive rows on the grid paper. In any of these rows squares that belong to M_w form a *strip* (i.e., quadrangle $1 \times \lambda$, $\lambda \geq 1$; values $\lambda = \lambda_s$ depend on the row number). For strips in two consecutive rows the *rightmost* square of the lower strip lies under the *leftmost* square of the upper strip.

There is a one-to-one correspondence between strip polygons M_w and words w of letters v and h: not only does M_w uniquely determine w, but vice versa as well. Therefore we can *identify* words w with polygons M_w and speak about properties of words and superwords geometrically - as properties of strip polygons.

Permitted and Forbidden Words.

We call a word w of letters v and h *permitted* if it is a subword of a certain superword $W_L \in S$ and *forbidden* otherwise.

Geometrically, if w is a *permitted* word then one can cross all squares of M_w by a single line L that contains no knots; if w is a *forbidden* word then there exists no such line.

0.3. Prove thaT $\{h\,v\,v\,v\,h\,v\,v\,h\}$ is a permitted word and hence there exists a superword $W = \{\ldots h\,v\,v\,v\,h\,v\,v\,h \ldots\} \in S$.

0.3.1. How many letters v can W have between *consecutive* letters h, i.e., letters h between which there are no other letters h? Is it true that either 2 or 3?

0.3.2. List all possible subwords of lengths 2, 3 and 4 in W.

0.3.3. Find two lines L and L^*, such that W_L and W_{L^*} contain subword $hvvvhvvh$ (as in 0.3).

0.3.4. Is it true that for all W_L and W_{L^*} sets of all subwords (a) of length 5; (b) of length 6; (c) of length 7 necessarily coincide?

0.4.1. Can it happen that all subwords of a certain superword W are permitted, but $W \notin S$?

0.4.2. Can W_L and W_{L^*} coincide while L and L^* do not?

0.4.3. Can the sets of all subwords of superwords W_L and W_{L^*} coincide, while W_L and W_{L^*} do not?

0.5.1. Erase one letter v in front of each h in $W \in S$ from 0.3.

Will the resulting superword $\{\dots hvvhvh\dots\}$ still belong to class S?

0.5.2. Erase one letter v in front of each h in any superword W from S.

Will the resulting superword still belong to S?

0.6. Insert one v in front of each h in any superword W from S.

Will the resulting superword still belong to S?

Classification of Superwords $W \in S$.

Ascribe $W = W_L \in S$ to S^h if line L is *slanting*: its slope is less than one; if instead L is *steep* (i.e., its slope is above one) ascribe $W = W_L \in S$ to S^v. Isolate superword $\{\dots v\,h\,v\,h\dots\}$, for which the slope is equal to one, to a special class E.

Ascribe superword $W \in S^h$ to S^h_n, $n = 1, 2, 3, \dots$, if one of the following holds:

a) between any two consecutive letters h in W there are n letters v,

b) between any two consecutive letters h in W there are either $n+1$ or n letters v.

Similarly, ascribe $W \in S^v$ to S^v_n, $n = 1, 2, 3, \dots$, if between any two consecutive letters v in W there are (a) n letters h, (b) either $n+1$ or n letters h.

Isolate $\{\dots v\,v\,v\,\dots\}$ and $\{\dots h\,h\,h\dots\}$ to subclasses S^h_∞ and S^v_∞.

0.7. Prove that the union of all S^h_n, $1 \le n \le \infty$, coincides with S^h and the union of all S^v_n, $1 \le n \le \infty$, coincides S^v.

Equivalently: prove that each superword $W \in S$ belongs to exactly one subclass:

$$\text{either } E, \text{ or } S^h_n \text{ or } S^v_n,\ 1 \le n < \infty, \text{ or } S^h_\infty \text{ or } S^v_\infty.$$

0.7.1. Interpret 0.7 geometrically in terms of strip polygons: what values can take lengths λ_s of strips of polygon M_w for permitted word w which is a subword of superword $W \in S^h$?

0.7.2. Give a periodic forbidden word w, for which all strips of M_w are of lengths 2 and 3.

0.8. Reorganization of superwords W, series of numbers a_j and continued fractions.

We take a word $W \in S$ and start to *reorganize* it, turning it successively into superwords W_j, $j \ge 1$: $W \to W_1 \to W_2 \to \dots$, and simultaneously constructing a *series of numbers* a_j according to the following rules.

Any superword W from S belongs to either E or S^h or S^v. Assume $W \in S^h$. Then, by 0.7, $W \in S^h_n$ for some n, $1 \le n \le \infty$.

If $n = \infty$ (i.e., if $W \in S^h_\infty$), no reorganization is made.

If $n = a_1 < \infty$ erase a_1 letters v in front of each h and call the resulting superword W_1. Clearly W_1 belongs to S^v (why?), and, therefore, belongs to S^v_n for some n.

If $n = \infty$ the reorganization is finished.

If $n = a_2 < \infty$, erase a_2 letters h in front of each v. We get W_2 and so on.

0.8.1. Let the slope of line L be equal to a *finite* continued fraction, for instance, one of the following:

$$\frac{11}{25}=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\frac{1}{2}}}},\quad \frac{7}{24}=\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3}}},\quad \frac{4}{19}=\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{3}}},\quad \frac{5}{8}=\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}.$$

For each of the above fractions find the series $a_1, a_2, \ldots, a_n$ for superword $W = W_L$ corresponding to line L.

0.8.2. It follows from 0.8.1. that for certain L reorganizations of $W = W_L$ may stop after finitely many steps: $W \to W_1 \to W_2 \to \cdots \to W_n$.

Classify *all* such L and for them state explicitly how series $a_1, a_2, \ldots, a_n$ are related to expansion of the slope k_L into continued fraction.

We will soon come back to studying relationship between continued fractions and series a_j but before we analyze some properties of superwords.

1. Class S_1: Carefully Mixed Words and Superwords.

Just for a change in the following problems we assume that superwords W consist of letters a and b. When relating to superwords from class S one can substitute $a \rightleftarrows h,\ b \rightleftarrows v$.

Let a word (or a superword) W be such that for every two its subwords U and V of the same length the number of letters a in U and in V is different by at most one (the same then applies to the number of letters b). We call such W *carefully mixed* words (superwords) and ascribe them to class S_1.

1.1. Give an example of carefully mixed superword.

Our immediate goal is to study the following two questions:

I. Is it true that every permitted word is carefully mixed?

II. Is it true that $S \subset S_1$?

The following exercises will help.

1.2.1. Is it true that every permitted word is a subword for infinitely many different superwords W_L from S?

1.2.2. Is it true that every permitted word is a subword of a superword W_L from S for a line L with rational slope k_L?

1.2.3. Is it true that superword W is carefully mixed if every its subword is carefully mixed?

1.3. Let a word (or a superword) W be splittable into subwords of the following two types:

ab^n (a letter a is followed by n letters b) and ab^{n+1} (a letter a is followed by $n+1$ letters b).

Perform the following *renaming*: substitute b for each ab^{n+1} and a for each ab^n. The resulting word (or superword) denote by W^*.

1.3.1. Is it true that after renaming permitted words go to permitted words and forbidden words go to forbidden words? What happens to strip polygons?

Hint: on the grid paper introduce coordinates x, y and consider mapping F sending point $\{x, y\}$ to $\{x - ny, -x + (n+1)y\}$.

1.3.2. Is it true that after renaming

words $W \in S_1$ go to $W^* \in S_1$, while words $W \notin S_1$ go to $W^* \notin S_1$?

1.4.1. Prove that every permitted word is carefully mixed.

1.4.2. Prove that every superword $W \in S$ is carefully mixed.

1.4.3. Give an example of carefully mixed superword $W \notin S$.

1.5. Is it true that in alphabet $\{a; b\}$ for every positive integers m and n there exists a periodic carefully mixed superword such that its period has m letters a and n letters b?

1.6.1. Does there exist a carefully mixed finite word that can not be a subword of a carefully mixed superword?

1.6.2. Is it true that any carefully mixed word is permitted?

1.7. Find all superwords W such that $W \in S_1$ and $W \notin S$.

1.8. Generalize the definition of carefully mixed words onto words over alphabets of more that two letters. Is it true that no carefully mixed word of letters a, b, c contains subword aa?

2. Class S_2: Superwords with Slowly Growing Subword Sets.

2.1. Let W be any superword of two letters and let $T_W(n)$ be the number of different subwords of length n in W. What is the maximum possible value of $T_W(n)$?

2.2. Prove that if $T_W(k) = T_W(k+1)$ for some k then superwordo W is periodic with period $T_W(k)$.

2.3. Let $T_W(k) < k+1$ for some k (compare to 2.1). Prove that superword W is periodic.

Therefore, minimum value of $T_W(n)$ for a nonperiodic word W is $n+1$. If this minimum is reached (see 2.4.2) and $T_W(k) = k+1$ for all k we call W *a superword with slowly growing* (when k increases) *subword set* and ascribe such W to class S_2.

2.4. Prove that for all superwords $W \in S$ inequality

$$T_W(k) \leq k+1$$

holds for all k.

2.4.1. Describe function $T_W(k)$ for superword $W = W_L$ when the slope k_L of line L is rational number m/n (for relatively prime m and n).

2.4.2. Let k_L be irrational. Prove that in this case $W = W_L$ satisfies $T_W(k) = k+1$ for all k, i.e., $W \in S_2$.

2.4.3. Give an example of superword W with slowly growing subword set that belongs neither to S nor to S_1.

2.4.4. Find all superwords $W \in S_2$ such that $W \in S_1$ and $W \notin S$.

Class S_2. Singular Words, Branching to the Left or to the Right.

By $\Pi_W(n)$ denote the set of all subwords of length n for a superword W.

2.5.1. Consider a superword $W \in S_2$. Erase the leftmost letter in each of the $n+1$ different subwords of length n belonging to $\Pi_W(n)$. Prove that each subword $U \in \Pi_W(n-1)$ will appear among $n+1$ resulting words of length $n-1$.

Therefore, all words $U \in \Pi_W(n-1)$ except one will appear once and only one subword — we call it *singular* or *branching to the left* — will appear twice.

The intuition behind the word 'branching': over the alphabet $\{v,h\}$ one can add both v and h to subword U from the left, while to nonsingular words one can add – *either* v *or* h.

Prove that singular *branching to the left* subwords U_n of length n form an embedded sequence $U_1 \subset U_2 \subset \cdots \subset U_n \subset U_{n+1} \subset \ldots$ (in other words, for each n singular U_n can be obtained by erasing the rightmost letter of U_{n+1}).

2.5.2. Similarly to 2.5.1. define *singular branching to the right* subwords V_n of a superword $W \in S_2$.

Prove that, likewise, singular *branching to the right* subwords V_n of length n form an embedded sequence $\dots \supset V_{n+1} \supset V_n \supset \dots \supset V_2 \supset V_1$.

2.5.3. Give an example of a superword $W \in S_2$ such that $U_n \neq V_n$ for all n.

2.5.4. Give an example of a superword $W \in S_2$ such that $U_n = V_n$ for infinitely many lengths n: $U_{n_k} = V_{n_k}$, $k = 1, 2, 3, \dots$.

2.6.1. Is it true that $W \in S_2$ if and only if for each n there exists exactly one branching to the left subword $U \subset W$ of length n?

2.6.2. Is it true that $W \in S_2$ if and only if for each n there exists exactly one branching to the right subword $V \subset W$ of length n?

2.7. Which of the following superwords, each combined from two periodic parts, belong to S_2?

$\{\dots vvv, hv^n hv^n hv^n \dots\}$, $n \geq 0$; $\{\dots hv^2hv^2hv^2, hv^5hv^5hv^5 \dots\}$;

$\{\dots hv\, hv\, hv\, hv,\ hvhvhvv\, hvhvhvv\, hvhvhvv \dots\}$;

$\{\dots hv\, hv\, hv\, hv,\ hvhvvhvv\, hvhvvhvv\, hvhvvhvv \dots\}$.

2.8.1. Find all superwords $W \in S_2$ such that $W \notin S_1$.

2.8.2. Suppose $W \in S_2$, $W \notin S_1$. Prove that for any subword $w \in W$ strip polygon M_w can be crossed by two half-lines with rational slopes (possibly depending on w).

2.8.3. Suppose $W \in S_2$, $W \notin S_1$. Is it true that there exist two rational numbers k_1 and k_2 such that for any subword $w \in W$ strip polygon M_w can be crossed by two half-lines with slopes k_1 and k_2? (with same k_1 and k_2 for all w).

Preparatory Problems: Lagrange and Hirzebruch fractions

The problems below are intended to accompany problems on superwords.

A. Geometry of Continued Fractions. Consider a quadrangle Π with sides p, q, where p and q – are mutually prime integers such that $1 < p < q$. Adjoin as many squares with side p to the smaller side as will fit inside the quadrangle (see the figure) and denote the number of them by a_1; therefore,

$$a_1 p < q < (a_1 + 1)p. \tag{1}$$

Cutting these squares off Π gives a new quadrangle Π_1 with sides $p_1 = q - a_1 p$, $q_1 = p$, where, by (1), $1 \leq p_1 < q_1$.

If $p_1 = 1$, assume $a_2 = q_1 = p$ and stop. If $p_1 < 1$, continue with Π_1 what we did with Π: adjoin as many squares with side p_1 as will fit inside Π_1 and denote the number of them by a_2. Cutting them off Π_1 gives quadrangle Π_2 with sides $p_2 = q_1 - a_2 p_1$, $q_2 = p_1$ and so on. The construction stops at step k if $p_k = 1$; in this case $a_k = q_k = p_{k-1} > 1$.

A0.1. Prove that

$$\frac{p}{q} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_k}}}}. \tag{2}$$

Numbers a_j in (2) are called *partial quotients* or *elements* of a continued fraction.

All a_j in (2), except a_n, can be any positive integers while $a_n > 1$.

A0.1. Alongside with finite one can consider infinite continued fractions with any integers a_0 and positive integers $a_n,\ n > 0$; Such an infinite continued fraction c is defined as *the limit* of finite continued fractions c_k when $k \to \infty$:

$$c = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = \lim c_k, \quad c_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_k}}}}.$$

Prove that $\lim c_k$ exists for any integer a_0 and positive integers $a_n,\ n > 0$.

A1.1. We have just given a geometric interpretation of partial quotients $a_1, \dots, a_k$ for the continued fraction expansion of a rational number p/q. Give a similar interpretation for an irrational number β expansion.

A1.2. Explain geometrically why $\beta = \sqrt{2} + 1$ or $\beta = (\sqrt{5} - 1)/2$ give *periodic* continued fractions.

A1.3. Prove that a number β gives a *periodic* continued fraction if and only if β is a *quadratic irrationality*, that is, β is a root of a quadratic equation with integer coefficients.

Explain when the result is a periodic fraction with period $\{a_1, \dots, a_p\}$ and when the period starts not from a_1 but from a_{k+1}, i.e., looks like $\{a_{k+1}, \dots, a_{k+p}\}$.

A2. The continued fractions introduces above are called Lagrange fractions[1]. Alongside with them we will be considering Hirzebruch continued fractions[2], finite or infinite:

$$\cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots - \cfrac{1}{b_k}}}}, \quad \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots}}}, \quad \text{все } b_j \geq 2. \tag{3}$$

Check the following examples:

$$\frac{4}{5} = \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}}}, \quad \frac{5}{8} = \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{2}}}, \quad \frac{3 - \sqrt{5}}{2} = \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{3 - \dots}}}. \tag{4}$$

Give a geometric interpretation of *elements* b_j for expansions (3).

In (3) and (4) only numbers less than one are expanded into Hirzebruch fractions. For numbers greater than or equal to one expansions look like

$$b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots - \cfrac{1}{b_k}}}}, \quad b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots}}}, \quad \text{все } b_j \geq 2. \tag{5}$$

[1] Joseph Louis Lagrange (1736-1813) – a great French mathematician.

[2] Friedrich Hirzebruch (born 1928) – a prominent modern German mathematician.

$$\text{For instance,}\quad \frac{5}{4}=2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2}}},\quad \frac{8}{5}=2-\cfrac{1}{3-\cfrac{1}{2}},\quad \frac{\sqrt{5}+3}{2}=3-\cfrac{1}{3-\cfrac{1}{3-\cfrac{1}{3-\dots}}}. \tag{6}$$

Remark. Condition *all* $b_j \geq 2$ in (3) and (5), guarantees *uniqueness* of Hirzebruch continued fraction expansion for any real number. If it were not imposed, one could *erroneously* bring about the following expansions:

$$\frac{5+\sqrt{5}}{2}=5-\cfrac{1}{1-\cfrac{1}{5-\cfrac{1}{1-\dots}}},\qquad \frac{5-\sqrt{5}}{2}=\cfrac{1}{1-\cfrac{1}{5-\cfrac{1}{1-\dots}}}. \tag{7}$$

Though equalities (7) are correct, they are *not* Hirzebruch expansions since the condition $b_j \geq 2$ is not satisfied.

A3.1. Give Hirzebruch expansions for numbers $1, (\sqrt{5}\pm 1)/2, (5\pm\sqrt{5})/2, \sqrt{2}$.

A3.2. Prove, under condition that *all* $b_j \geq 2$, uniqueness of Hirzebruch expansion for any positive real number.

A3.3. Solve problem A1.3 for Hirzebruch fractions.

Convergents of a Continued Fraction. Introduce the following (more concise) notation for infinite Lagrange and Hirzebruch continued fractions with elements a_j and b_j:

$$[a_0; a_1, a_2, \dots], \quad [[b_0; b_1, b_2, \dots]]. \tag{8}$$

Finite continued fractions are denoted as follows:

$$[a_0; a_1, a_2, \dots, a_k], \quad [[b_0; b_1, b_2, \dots, b_k]]. \tag{9}$$

If $a_0 = 0$ or $b_0 = 0$ in (8) or (9), we write

$$[a_1, a_2, \dots], \quad [[b_1, b_2, \dots]]; \quad [a_1, a_2, \dots, a_k], \quad [[b_1, b_2, \dots, b_k]].$$

Finite fractions

$$[a_0;], [a_0; a_1], [a_0; a_1, a_2], \dots \quad \text{и} \quad [[b_0;]], [[b_0; b_1]], [[b_0; b_1, b_2]], \dots \tag{10}$$

which approximate fractions (8) and (9), are called *convergents*. The following fractions, for instance, are convergents for Lagrange expansion $[1; 1, 1, \dots]$ of $(\sqrt{5}+1)/2$:

$$[1;] = 1/1,\ [1;1] = 2/1,\ [1;1,1] = 3/2,\ [1;1,1,1] = 5/3,\ \dots \tag{11}$$

while convergents for Hirzebruch expansion $[[3; 3, 3, \dots]]$ of $(\sqrt{5}+3)/2$ are

$$[[3;]] = 3/1,\ [[3;3]] = 8/3,\ [[3;3,3]] = 21/8,\ [[3;3,3,3]] = 55/21,\ \dots \tag{12}$$

By (12), convergents for expansion of $(\sqrt{5}+1)/2 = [[2; 3, 3, \dots]]$ would be

$$[[2;]] = 2/1,\ [[2;3]] = 5/3,\ [[2;3,3]] = 13/8,\ [[2;3,3,3]] = 34/21,\ \dots \tag{13}$$

A4.1. How are convergents (11)–(13) related to Fibonacci sequence

$$f_0 = 0,\ f_1 = 1, \quad f_{k+1} = f_k + f_{k-1}?$$

A4.2. Find and compare to each other Lagrange and Hirzebruch convergents of $\frac{\sqrt{5}-1}{2}$.

B. Nose Stretching Algorithm. For a ray $y = \beta x$ going from the origin there exists a connection between continued fraction expansion of its slope β and convex hulls of the sets of all the knots *below* and *above* the ray.

Consider broken lines $\underline{L}$ and $\overline{L}$ that bound these convex hulls. Famous russian mathematician Boris Delone calls $\underline{L}$ and $\overline{L}$ *noses stretching along ray* $y = \beta x$.

B0. Show that vertices of $\underline{L}$ and $\overline{L}$ are points x_k, y_k, where y_k/x_k are convergents for Lagrange expansion of the slope β.

B1. Can you give a geometric interpretation for Hirzebruch expansion of the slope β? Consider cases $\beta < 1$ and $\beta > 1$.

E-mail address: kanel@mccme.ru M.L@gerver.mccme.ru