Наборы кратных сумм

К. Кохась

August 8, 2002

Условия задач

Let $A = \{a_1, a_2, \dots, a_n\}$ be a set of arbitrary natural numbers (some of a_i may be equal). We define a collection of k-sums as a set of numbers of the form $a_{i_1} + a_{i_2} + \ldots + a_{i_k}$, where $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$. Denote this set $A^{(k)}$. For instance, $A^{(3)} = \{4, 9, 10, 10\}$ for $A = \{1, 1, 2, 7\}$.

The $\diamond$ labels problems that will be considered and solved at the intermediate finish.

1 Pairwise sums

$\diamond$ **1.1.** Given numbers a_1, a_2, a_3, a_4, a_5, we can form 10 pairwise sums x_1, x_2, ... , x_{10}. Prove that if we know numbers x_1, x_2, ... , x_{10} (however, for any number, we are not aware what are the numbers of which it is the sum) we can recover the numbers a_1, a_2, a_3, a_4, a_5.

$\diamond$ **1.2.** It is known that, for a set A, its pairwise sums is equal to $\{2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6\}$. Find the sum of squares of the elements of A.

$\diamond$ **1.3.** а) Find an example of two sets of four numbers (equal numbers are allowed) that have coinciding 2-sums (pairwise sums).

б) Let $b_1 < b_2 < b_3 < b_4 < b_5 < b_6$ be a collection of 2-sums of some set of four numbers. Prove that there exists one more collection of numbers with the same set of 2-sums.

$\diamond$ **1.4.** Prove that for $n = 2^k$, $k = 1, 2, \dots$, there exist two collections of n numbers that have coinciding collections of pairwise sums.

$\diamond$ **1.5.** Prove that for $n \neq 2^k$ the set of n numbers is uniquely defined by its collection of pairwise sums.

1.6. а) Find an example of three sets of numbers that have coinciding collections of 2-sums.

б) Prove that there exist no three sets of four numbers that have coinciding collections of 2-sums.

в) Prove that there exist no four sets of eight numbers that have coinciding collections of 2-sums.

1.7. A boy Sasha chooses 3 real numbers and writes down all their pairwise sums, pairwise sums taken with the minus sign, and also pairwise differences (totally 12 numbers). His girlfriend Nadya wants to find out the modules of the initial numbers. Is it always possible (discuss a constructive procedure)?

1.8. Prove that for a collection $\{a_1, a_2, \dots\}$ containing n different numbers the number of different sums of the form $a_i + a_j$ may be equal to any number from the interval $2n - 1, C_n^2$.

2 Sums of higher order

2.1. а) Prove that for any nonnegative integer m the numbers 2, 2^2, 2^3, ... , $2^{2 \cdot 3^{m-1}}$ have different residues modulo 3^m.

б) Prove that for natural p the equation

$$n^2 - (2^p + 1)n + 2 \cdot 3^{p-1} = 0 \tag{1}$$

has a solution in natural numbers (with respect to n) only if $p = 1, 2, 3, 5, 9$.

2.2. Prove that the collection of n numbers is uniquely determined by its collection of 3-sums if $n \neq 3, 6, 27, 486$.

$\diamond$ **2.3.** а) Prove that for any $k \in \mathbb{N}$ the collection of $2k$ numbers is, in general, not determined uniquely by its collection of k-sums.

б) Prove that a collection of n numbers is uniquely determined by its k-sums if and only if it is uniquely determined by its $(n - k)$-sums.

2.4. a) Is it true that a collection of 27 numbers is uniquely determined by its 3-sums?

б) Is it true that a collection of 486 numbers is uniquely determined by its 3-sums?

2.5. Prove that for natural p the equation

$$n^3 - (3 \cdot 2^{p-1} + 3)n^2 + (2 + 3 \cdot 2^{p-1} + 2 \cdot 3^p)n - 6 \cdot 4^{p-1} = 0 \qquad (2)$$

has a solution in natural numbers (with respect to n) only if $p \leqslant 7$. Then, possible solutions are $n = 1, 2, 3, 4, 8, 12$.

2.6. Is it true that there exist no four collections of 6 numbers each which have the coinciding collections of 3-sums?

2.7. Prove that a collection of 12 numbers is uniquely determined by its 4-sums.

2.8. Prove that if $(k-1)!k^{n-1} \not\vdots\, n$ (for instance this happens if k is less than the maximal prime divisor of n) then a collection A of n numbers is uniquely determined by the collection $A^{(k)}$.

2.9. Given two collections A and B of n numbers each have coinciding sets of k-sums. It is known that elements of the collection B are in “a general position”, which means that all sums of the form

$$\pm b_{i_1} \pm b_{i_2} \ldots \pm b_{i_m}, \quad \text{где} \quad 1 \leqslant m \leqslant n,\ 1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant n$$

are pairwise different. Prove that either $n = k$ or $n = 2k$. In the latter case the collections A and B are symmetric to each other with respect to some point at the real axis.

3 Addition after the intermediate finish

For an arbitrary collection of numbers $X = \{x_1, x_2, \ldots, x_n\}$ denote

$$s_p(X) = x_1^p + x_2^p + \ldots + x_n^p\,.$$

Moreover, if $1 \leqslant k \leqslant n$ then for every set $(p_1, p_2, \ldots, p_k)$ of nonnegative integers denote

$$s_{p_1,p_2,\ldots,p_k} = \sum x_{i_1}^{p_1} x_{i_2}^{p_2} \ldots x_{i_k}^{p_k}\,,$$

where the sum runs over all ordered sets $\{i_1, i_2, \ldots, i_k\} \subset X$.

3.1. Several auxiliary combinatorial facts.

a) Prove that there are exactly $\sum\limits_{i=1}^{r} (-1)^{r-i} C_r^i i^k$ ways to put k different photographs into $r \leqslant k$ different envelopes such that every envelope would contain at least one photograph.

б) Prove that $C_n^{s-q} = \sum\limits_{v=q}^{s} C_{n-v}^{s-v} C_{v-1}^{q-1}$.

в) Let the set $\{p_1, p_2, \ldots, p_k\}$ contains exactly t zeroes. Prove that

$$s_{p_1,p_2,\ldots,p_k} = (n-k+1)(n-k+2)\ldots(n-k+t) \cdot (-1)^{k-t-1} \cdot (k-t-1)!s_p + F\,,$$

where $p = p_1 + \ldots + p_k$ and the expression F involves only the terms $s_1, \ldots, s_{p-1}$.

3.2. Prove that

$$s_p(A^{(k)}) = f(n,k)s_p(A) + G\,,$$

where the term G depends on n, k, $s_q(A)$, $1 \leqslant q < p$, but does not depend on $s_p(A)$ and

$$f(n,k) = \sum_{i=1}^{k} (-1)^{i-1} i^{p-1} C_n^{k-i}\,.$$

3.3. Find a simple formula for $f(n,k)$ when $p = 1$.

3.4. Find any $k \geqslant 5$ and $n > 2k$ such that $f(n,k) = 0$ for some p.

3.5. Is the set of pairs of natural numbers (m,n) for which

$$15(8m^2 - 8mn + n^2 + 3n)^2 - 7n^4 + 38n^3 - 47n^2 - 32n = 0$$

finite or infinite? Here is the list of solutions (n,m) for $n < 400$ or $m < 400$: (0,0), (2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (12,4), (12,8), (32,2), (32,10), (32,22), (32,30). There is one more pair of integer solutions: (−1,−1), (−1,0).

3.6. For an arbitrary set of numbers A define a collection $A^{[k]}$ of its k-sums with repetitions as a collection of all numbers of the form $a_{i_1} + a_{i_2} + \ldots + a_{i_k}$, where $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k \leqslant n$. Prove that for any k the collection A is uniquely determined by $A^{[k]}$.

Solutions

1 Pairwise sums

1.1. The problem is of rich olympiad history. It was presented at the Moscow city olympiad in 1962 (2nd tour, 8th grade, see [2], problem 25.29. A similar problem with concrete numbers was presented at the St. Petersburg olympiad in 1968 (7th grade). A bit earlier it was published in the American Mathematical Monthly ([11], problem E1248a, by L. Moser), and before that in the Newsletter of Canad. Math. Congress (December, 1955).

Note that $\sum x_i = 4\sum a_j$. One can assume that x_i's and a_j's are numbered in the increasing order. Then $x_1 = a_1 + a_2$, $x_2 = a_1 + a_3$, $x_9 = a_3 + a_5$, $x_{10} = a_4 + a_5$. Hence $a_3 = \frac{1}{4}\sum x_i - x_1 - x_{10}$, $a_1 = x_2 - a_3$, $a_5 = x_9 - a_3$, $a_2 = x_1 - a_1$, $a_4 = x_{10} - a_5$.

1.2. Answer: $25\frac{1}{2}$. Let $s_1(A)$ be the sum of all numbers in A, $s_2(A)$ be the sum of squares of all numbers in A. Then the sum of numbers in $A^{(2)}$ is equal to

$$s_1(A^{(2)}) = \sum_{i<j}(a_i + a_j) = (n-1)s_1(A)\,.$$

Hence $s_1(A) = \frac{1}{n-1}s_1(A^{(2)})$. The sum of squares of numbers in $A^{(2)}$ is equal to

$$s_2(A^{(2)}) = \sum_{1\leqslant i<j\leqslant n}(a_i+a_j)^2 = \frac{1}{2}\sum_{i\neq j}(a_i^2+a_j^2+2a_ia_j) = \frac{1}{2}\left(\sum_{i,j=1}^{n}(a_i^2+a_j^2+2a_ia_j)-\sum_{j=1}^{n}(2a_j)^2\right) = (n-2)s_2(A)+s_1(A)^2,$$

and so $s_2(A) = \frac{1}{n-2}\left(s_2(A^{(2)}) - s_1(A)^2\right)$.

In our case the set A consists of 6 numbers, i.e. $n = 6$; $s_1(A^{(2)}) = 55$, $s_2(A^{(2)}) = 223$. Hence $s_1(A) = 11$, $s_2(A) = \frac{1}{4}(223 - 121) = 25\frac{1}{2}$.

1.3. a) For example $\{1, 1, 2, 4\}$ and $\{0, 2, 3, 3\}$.

б) This problem (together with the solution) is from [11], problem E1248b, by L. Moser. Let a_1, a_2, a_3, a_4 be the numbers considered, $S = (a_1 + a_2 + a_3 + a_4)/2$. The set $\{S - a_1, S - a_2, S - a_3, S - a_4\}$ has the same set of 2-sums. It does not coincide with the first one. Indeed, the numbers b_3 and b_4 are equal to the numbers $a_1 + a_4$ and $a_2 + a_3$ (perhaps in reverse order). As $b_3 \neq b_4$, we see that $S - a_4 \neq a_1$. Thus the minimal numbers in two sets above are not equal and the sets are different.

1.4. This problem and the following one were presented in 1987 at the olympiad of the school No. 239, St. Petersburg ([4], с. 297, задача 79) at the stage "Grand-prix" consisting of researching problems. As to this very problem, the organizers overestimated capabilities of school students. One can find solutions both for this problem and the problem 1.5 in articles [12], [3].

Let us construct the sets A_n and B_n (each consisting of $n = 2^k$ numbers) with coinciding sets of 2-sums by induction on k.

The basis of induction $k = 1$, $n = 2$. $A_2 = \{0, 3\}$, $B_2 = \{1, 2\}$.

The step of induction. Assume that the sets $|A_m| = |B_m| = 2^m$ with coinciding 2-sums are already defined. Choose a number d such that the sets

$$A_{m+1} = A_m \cup (B_m + d), \qquad B_{m+1} = B_m \cup (A_m + d)$$

do not coincide. Let us prove that the sets of 2-sums of the sets constructed coincide. For this we will check that for each pair $(\alpha_1, \alpha_2) \subset A_{m+1}$ there exists a pair $(\beta_1, \beta_2) \subset B_{m+1}$, and that this is a one-to-one correspondence.

1st case. α_1, $\alpha_2 \in A_m$. Take a pair (β_1, β_2) such that β_1, $\beta_2 \in B_m$.

2nd case. α_1, $\alpha_2 \in B_m + d$. Take a pair (β_1, β_2) such that β_1, $\beta_2 \in A_m + d$.

3rd case. $\alpha_1 \in A_m$, $\alpha_2 \in B_m + d$. Take a pair $(\alpha_1 + d, \alpha_2 - d)$; $\alpha_1 + d \in A_m + d$, $\alpha_2 - d \in B_m$.

1.5. Let A be a set of $n \neq 2^k$ numbers. We check by induction on p that the standard symmetric polynomials (power sums)

$$s_p(x_1, x_2, \dots) = x_1^p + x_2^p + \dots\,,$$

for the set A are determined by the power sums of the set $A^{(2)}$.

First of all, $s_1(A^{(2)}) = C_{n-1}^1 s_1(A)$, hence $s_1(A) = (C_{n-1}^1)^{-1}s_1(A^{(2)})$.

Then

$$s_p(A^{(2)}) = \sum_{i<j}(a_i+a_j)^p = \frac{1}{2}\sum_{i\neq j}(a_i+a_j)^p = \frac{1}{2}\sum_{i\neq j}\sum_{r=0}^{p} C_p^r a_i^r a_j^{p-r} = \frac{1}{2}\sum_{r=0}^{p} C_p^r \sum_{i\neq j} a_i^r a_j^{p-r} =$$
$$= \frac{1}{2}\sum_{r=0}^{p} C_p^r\big(s_r(A)s_{p-r}(A) - s_p(A)\big) = \frac{1}{2}\sum_{r=1}^{p-1} C_p^r s_r(A)s_{p-r}(A) + (n-2^{p-1})s_p(A)\,, \tag{3}$$

in the last equality we use that $s_0(A)=n$. By induction hypothesis all the terms in the first sum of the right hand side are determined by the power sums of the set $A^{(2)}$. Hence $s_p(A)$ is determined by the power sums of the set $A^{(2)}$ if $n\neq 2^{p-1}$.

We checked that the power sums $s_1(A)$, $s_2(A)$, … , $s_n(A)$ are determined by the set $A^{(2)}$. Therefore the initial numbers, i.e. the set A, are also determined by the set $A^{(2)}$. Indeed, let

$$\sigma_k(A) = \sum_{1\leqslant i_1<\dots<i_n\leqslant n} a_{i_1}a_{i_2}\dots a_{i_n}$$

denote kth elementary symmetric function of the set A. Then it is easy to check the following Newton's formulae:

$$n\sigma_n = \sigma_{n-1}s_1 - \sigma_{n-2}s_2 + \sigma_{n-3}s_3 - \dots + (-1)^n\sigma_1 s_{n-1}\,.$$

It is clear that the numbers $\sigma_1(A)$, $\sigma_2(A)$, … can be determined one by one due to these formulae if the numbers $s_1(A)$, $s_2(A)$, … are known. To restore the set A we need to solve the equation

$$x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \dots + (-1)^{n-2}\sigma_{n-2}x^2 + (-1)^{n-1}\sigma_{n-1}x + (-1)^n\sigma_n = 0\,. \tag{4}$$

Note that the problem 1.2 was an essential hint for inventing this solution.

1.6. a) Of course, we'll consider the case $n>2$. The next problem claims that there do not exist such sets containing 4 numbers each. Such sets of 8 elements do exist ([10]):

$$\{\pm1,\pm9,\pm15,\pm19\},\quad \{\pm2,\pm6,\pm12,\pm22\},\quad \{\pm3,\pm7,\pm13,\pm21\}.$$

In general [9], if $X_1 = \{a,b,c,d\}$ is an arbitrary set, then the sets $X = X_1\cup(-X_1)$, $Y = Y_1\cup(-Y_1)$, $Z = Z_1\cup(-Z_1)$, where

$$Y_1 = \left\{\tfrac{1}{2}(-a+b+c+d), \tfrac{1}{2}(a-b+c+d), \tfrac{1}{2}(a+b-c+d), \tfrac{1}{2}(a+b+c-d)\right\},$$
$$Z_1 = \left\{\tfrac{1}{2}(a+b+c+d), \tfrac{1}{2}(a+b-c-d), \tfrac{1}{2}(a-b+c-d), \tfrac{1}{2}(a-b-c+d)\right\},$$

have coinciding sets of 3-sums. In the previous example $X_1 = \{-1,9,15,-19\}$.

б) We follow [9]. Let A be a set of 4 numbers. Without loss of generality we can assume that $s_1(A)=0$ (we can translate our set). Use the formulae (3) for $n=4$:

$$\begin{aligned}
s_1(A^{(2)}) &= 0\\
s_2(A^{(2)}) &= 2s_2(A)\\
s_3(A^{(2)}) &= 0\\
s_4(A^{(2)}) &= -4s_4(A) + 3s_2^2(A)\\
s_5(A^{(2)}) &= -12s_5(A) + 10s_2(A)s_3(A)\\
s_6(A^{(2)}) &= -24s_6(A) + 15s_2(A)s_4(A) + 10s_3^2(A)
\end{aligned}$$

We see that the values of $s_2(A)$ and $s_4(A)$ are determined by sums $s_p(A^{(2)})$. Let us check that the last equation is quadratic with respect to $s_3(A)$. The expression $s_6(A)$ turns out to be a polynomial in $s_1(A)=0$, $s_2(A)$, $s_3(A)$, $s_4(A)$, and the degree of this polynomial with respect to $s_3(A)$ is at most 2.

To be more precise,

$$s_6 = \frac{3}{4}s_4s_2 + \frac{3}{4}s_4s_1^2 + \frac{1}{3}s_3^2 + 0s_3s_2s_1 - \frac{2}{3}s_3s_1^3 - \frac{1}{8}s_2^3 - \frac{3}{8}s_2^2s_1^2 + \frac{3}{8}s_2s_1^4 - \frac{1}{24}s_1^6 = \tag{5}$$
$$= \frac{3}{4}s_4s_2 + \frac{1}{3}s_3^2 - \frac{1}{8}s_2^3. \tag{6}$$

The equality (5) is a general fact that requires some machinery like theory of symmetric functions or a computer. But the equality (6) that we need (it is obtained from the previous formula by a substitution $s_1 = 0$) can be easily discovered.

So $s_3(A)$ has at most 2 different values. Therefore there exist at most 2 sets with the same set of 2-sums. The example of the general sets with coinciding sets of 2-sums is given by the formula

$$X = \{a, b, c, d\} \quad \text{and} \quad Y = \left\{\tfrac{1}{2}(-a+b+c+d), \tfrac{1}{2}(a-b+c+d), \tfrac{1}{2}(a+b-c+d), \tfrac{1}{2}(a+b+c-d)\right\}.$$

в) The solution is similar to the previous one, but much more technical. :(

1.7. The problem is suggested by А. Храбров and Н. Гуревич. Answer: Yes, initial numbers can always be determined. Let a_1, a_2, a_3 be modules of Sasha's numbers. Add to the set of 12 sums/differences three zeros. The set obtained is the set of 2-sums for the set $\{a_1, a_2, a_3, -a_1, -a_2, -a_3\}$. We already know that the set of 6 numbers is determined by the set of 2-sums. Though Nadya may not be aware about details, she can determine the numbers as follows. Starting from values of $s_1, s_2, \dots, s_6$ for Sasha's set she can calculate symmetric functions $\sigma_1, \sigma_2, \dots, \sigma_6$ for Sasha's set and then the set itself as a set of roots of the (bicubic) equation (4). To this end she can use the Cardano formula.

1.8. It is mentioned in [7] as an example of a trivial question.

2 Sums of higher order

Selfridge J.L. and Straus E.G. were the first to show the importance of the equations (1), (2) in the problem about 3- and 4-sums. ([12], 1958). They had solved these equations and proved that the set is not determined by 3- or 4 sums only if n is a root of these equations. Fomin and Izhboldin [3] independently discovered this fact in the beginning of 90th and gave examples of pairs of sets with coinciding 3-sums for $n = 27$ and $n = 486$.

2.1. а) It is sufficient to check that the numbers $2^{2\cdot 3^{m-2}} - 1 = \left(2^{3^{m-2}} - 1\right)\left(2^{3^{m-2}} + 1\right)$ and $\left(2^{3^{m-1}} - 1\right)$ are not multiples of 3^m. The expressions in the brackets, where the unit is subtracted, are not divisible even by 3. Prove by induction on m that $2^{3^{m-2}} + 1$ is divisible by 3^{m-1}, but not by 3^m. Indeed,

$$2^{3^{m-2}} + 1 = \left(2^{3^{m-3}} + 1\right)\left(2^{2\cdot 3^{m-3}} - 2^{3^{m-3}} + 1\right).$$

The first factor here is divisible by 3^{m-2} and not by 3^{m-1} by induction hypothesis. The second factor is divisible by 3 and not divisible by 9.

б) [12] Write the equation in the form

$$n + \frac{2\cdot 3^{p-1}}{n} = 2^p + 1.$$

Then, by Vieta theorem, n and $\frac{2\cdot 3^{k-1}}{n}$ are the roots of our quadratic equation, hence they have a form $n_1 = 2^a 3^b$, $n_2 = 2^{1-a}3^{p-b-1}$, where $a = 0$ or 1. We can assume that $n_1 \leqslant n_2$, then $b \leqslant p - b - 1$ and

$$2^p \equiv -1 \pmod{3^b}.$$

Since 2 is a primitive root modulo 3^b,

$$p \equiv 3^{b-1} \pmod{2\cdot 3^{b-1}}. \tag{7}$$

Then $3^{p-b-1} \leqslant n_2 < 2^p < 3^{2p/3}$, and therefore $p < 3(b+1)$. Now (7) leads to

$$3^{b-1} \leqslant p < 3(b+1).$$

Hence $b < 4$.

If $b = 3$ then $p \equiv 9 \pmod{18}$ and $p < 12$, so $p = 9$. If $b = 2$ then $p \equiv 3 \pmod 6$ and $p < 9$, so $p = 3$. If $b = 1$ then $p \equiv 1 \pmod 2$ and $p < 6$, so $p = 1, 3, 5$. If $b = 0$ then $p < 3$.

These ares the corresponding quadratic polynomials.

$$\begin{array}{llll} p=1 & \dfrac{1}{2}(n-1)(n-2), & p=2 & \dfrac{1}{2}(n-2)(n-3), \\ p=3 & \dfrac{1}{2}(n-3)(n-6), & p=5 & \dfrac{1}{2}(n-6)(n-27), \\ p=9 & \dfrac{1}{2}(n-27)(n-486). & & \end{array} \tag{8}$$

2.2. [12, 3]. Let A be initial set of n numbers, $A^{(3)}$ be the set of its 3-sums, and $s_p(x_1, x_2, \dots)$ be the power sums (as in problem 1.5). Then

$$6s_k(A^{(3)}) = \sum_{i_1,i_2,i_3=1}^{n} (x_{i_1}+x_{i_2}+x_{i_3})^k - 3\sum_{i_1,i_2=1}^{n}(2x_{i_1}+x_{i_2})^k + 2\sum_{i=1}^{n}(3x_i)^k =$$

$$= \sum_{i_1,i_2,i_3=1}^{n} \sum_{\ell_1+\ell_2+\ell_3=k} \frac{k!}{\ell_1!\ell_2!\ell_3!} x_{i_1}^{\ell_1} x_{i_2}^{\ell_2} x_{i_3}^{\ell_3} - 3\sum_{i_1,i_2=1}^{n}\sum_{\ell=0}^{k} C_k^\ell (2x_{i_1})^\ell x_{i_2}^{k-\ell} + 2\cdot 3^k \sum_{i=1}^{n} x_i^k =$$

$$= \sum_{\ell_1+\ell_2+\ell_3=k} \frac{k!}{\ell_1!\ell_2!\ell_3!} s_{\ell_1}s_{\ell_2}s_{\ell_3} - 3\sum_{\ell=0}^{k} C_k^\ell 2^\ell s_\ell s_{k-\ell} + 2\cdot 3^k s_k\,.$$

Canceling by 3 and applying the equality $s_0 = n$ we get the coefficient of $s_k(A)$ in the right hand side. It is equal to

$$f(n,k) = n^2 - (2^k+1)n + 2\cdot 3^{k-1}\,. \tag{9}$$

Hence the right hand side of the equality is of the form $f(n,k)s_k(A) + G$, where G is an expression depending on n and $s_i(A)$, $i < k$. As we have proved in the previous problem $f(n,k) \neq 0$ for $n \neq 3, 6, 27, 486$. Therefore, as in the problem 1.5, we obtain that the power sums of the set A are determined by the power sums of the set $A^{(3)}$. So in this case the set A is uniquely determined by $A^{(3)}$.

2.3. a) [3], [9]. For every set $A = \{a_1, a_2, \dots, a_n\}$ let us calculate its arithmetic mean $\bar{a} = (a_1 + a_2 + \ldots + a_n)/n$ and construct the symmetric set

$$A' = \{2\bar{a} - a_1, 2\bar{a} - a_2, \dots, 2\bar{a} - a_n\}\,.$$

We call the set A *nonsymmetric* if $A \neq A'$. If the sets of m-sums A and A' coincide, we call the set A *m-selfadjoint.*

Prove that if the set A of $2k$ numbers is nonsymmetric, then the sets A and A' have the same sets of k-sums. Indeed, for every k elements $a_{i_1}, \dots, a_{i_k}$ of the set A take those k elements $b_{j_1}, \dots, b_{j_k}$ of the set A', for which the set of indices $\{j_1, \dots, j_k\}$ is a complement of the set of indices $\{i_1, \dots, i_k\}$ in the whole set $\{1, 2, \dots, 2k\}$. Then

$$b_{j_1} + \ldots + b_{j_k} = (2\bar{a} - a_{j_1}) + \ldots + (2\bar{a} - a_{j_k}) = 2k\bar{a} - (a_{j_1} + \ldots + a_{j_k}) = a_{i_1} + \ldots + a_{i_k}\,.$$

б) If two sets have the same sets of k-sums, then the sums of numbers in these sets are equal, as we have proved in the problem 1.5. Let S be this sum. Then the set of k-sums and $(n-k)$-sums can be obtained from each other by the symmetry with respect to the number $S/2$.

2.4. The answer is "no" in both parts of the problem.

The next examples are from [3]. It is sufficient to find nonsymmetric 3-selfadjoint sets of 27 and 486 elements. Here you are:

$$n = 27: \quad A = \{0^5, 1^{10}, 2^{10}, 3^2\}\,, \qquad n = 486: \quad B = \{0^{22}, 1^{176}, 2^{231}, 3^{56}, 4^1\}\,.$$

Let us check for example that the set A is a 3-selfadjoint. Indeed, it is easy to check that it has the following set of 3 sums

$$A^{(3)} = \{0^{10}, 1^{100}, 2^{325}, 3^{640}, 4^{725}, 5^{640}, 6^{325}, 7^{100}, 8^{10}\}.$$

For example the number 3 we can get as a sum $0+0+3$ (20 ways), $1+1+1$ ($C_{10}^3 = 120$ ways) or $0+1+2$ (500 ways); totally 640 ways. Furthermore as we see, the set of 3-sums is symmetric with respect to 4, and the arithmetic mean of numbers in A is equal to $4/3$. Hence the set A', that are symmetric to A, has the same set of 3-sums. Indeed,

$$(2\cdot\frac{4}{3} - a_1) + (2\cdot\frac{4}{3} - a_2) + (2\cdot\frac{4}{3} - a_3) = 8 - (a_1 + a_2 + a_3)\,.$$

More examples. Next examples and the way to find them are from [6]. The solution of the problem 2.2 shows that we can try to recover the set A from the set $A^{(3)}$ by finding consequently the expression of $s_k(A)$ via power sums $A^{(3)}$. This will be always successful if the coefficient $f(n,k)$ in the formula (9) is not equal to 0. From the formulae (8) we see that the minimal k such that $f(n,k) = 0$ is $k = 5$.

Let us try to find a set A of 27 numbers that is nonsymmetric with respect to zero and such that the symmetric set $-A$ would have the same set of 3-sums. If it would be successful then the relations $s_k(A^{(3)}) = s_k\big((-A)^{(3)}\big)$

had meant that for odd k $s_k(A^{(3)}) = 0$ and therefore $s_1(A) = s_3(A) = 0$. To simplify our efforts let us seek a set containing its elements with repetitions. Let, say, we have three numbers only, the number a_i has the multiplicity n_i, $i = 1, 2, 3$. Then we have the following system of equations

$$\begin{aligned} n_1 + n_2 + n_3 &= 27 \\ n_1a_1 + n_2a_2 + n_3a_3 &= 0 \\ n_1a_1^3 + n_2a_2^3 + n_3a_3^3 &= 0 \end{aligned}$$

Let us try to take an arithmetic progression for a_n. Some experiments with the system show that the difference of the progression should be divisible by 3. Let us take $a_1 = -4$, $a_2 = -1$, $a_3 = 2$, then $n_1 = 1$, $n_2 = 16$, $n_3 = 10$. The set $A = \{-4, -1^{16}, 2^{10}\}$ have the symmetric set $A^{(3)}$:

$$A^{(3)} = \{-6^{120}, -3^{720}, 0^{1245}, 3^{720}, 6^{120}\}\,.$$

These are more examples: $\{-5^2, -2^{10}, 1^10, 4^5\}$ (this example differs from the example in the beginning of solution by scaling and translation), $\{-7^1, -4^5, -1^{10}, 2^6, 5^5\}$, $\{-8^2, -5^4, -2^6, 1^8, 4^3, 7^4\}$ and even

$$\{-14^2, -11^1, -8^3, -5^3, -2^3, 1^6, 4^2, 7^3, 10^1, 13^3\}\,.$$

Similarly, starting from the progression $a_1 = -7$, $a_2 = -4$, $a_3 = -1$, $a_4 = 2$, $a_5 = 5$ and the equations

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 &= 486 \\ (-7)n_1 + (-4)n_2 + (-1)n_3 + 2n_4 + 5n_5 &= 0 \\ (-7)^3n_1 + (-4)^3n_2 + (-1)^3n_3 + 2^3n_4 + 5^3n_5 &= 0 \\ (-7)^5n_1 + (-4)^5n_2 + (-1)^5n_3 + 2^5n_4 + 5^5n_5 &= 0 \\ (-7)^7n_1 + (-4)^7n_2 + (-1)^7n_3 + 2^7n_4 + 5^7n_5 &= 0\,, \end{aligned}$$

we will find a solution $n_1 = 1$, $n_2 = 56$, $n_3 = 231$, $n_4 = 176$, $n_5 = 22$. Yes, 7^7, what do you think. So we have found the set $B = \{-7, -4^{56}, -1^{231}, 2^{176}, 5^{22}\}$. To check that this example is valid we can write down the set of 3-sums:

$$B^{(3)} = \{-15^{1540}, -12^{40656}, -9^{392161}, -6^{1800568}, -3^{4358893}, 0^{??}, 3^{4358893}, 6^{1800568}, 9^{392161}, 12^{40656}, 15^{1540}\}$$

(the multiplicity of zero is not essential here).

2.5. [3], [12] Factor the given polynomials for $p \leqslant 7$:

$p = 1$	$\frac{1}{6}(n-1)(n-2)(n-3)$,	$p = 2$	$\frac{1}{6}(n-2)(n-3)(n-4)$,
$p = 3$	$\frac{1}{6}(n-3)(n-4)(n-8)$,	$p = 4$	$\frac{1}{6}(n-4)(n^2-23n+96)$,
$p = 5$	$\frac{1}{6}(n-8)(n^2-43n+192)$,	$p = 6$	$\frac{1}{6}(n-12)(n^2-87n+512)$,
$p = 7$	$\frac{1}{6}(n-8)(n^2-187n+3072)$,		

Further we will assume that $p \geqslant 8$. Write the equation in the form

$$n\bigl(n - (3\cdot 2^{p-1} + 3)\bigr) + (2 + 3\cdot 2^{p-1} + 2\cdot 3^p) = \frac{6\cdot 4^{p-1}}{n}\,. \tag{10}$$

It is easy to check that $2 + 3\cdot 2^{p-1} + 2\cdot 3^p$ is not divisible by 16. Then either n or $6\cdot 4^{p-1}/n$ is not divisible by 16.

1st case. n is not divisible by 16. Since $6\cdot 4^{p-1}$ is divisible by n, then either $n = 2^m$ or $n = 3\cdot 2^m$. Since n is not divisible by 16 then $n \in \{1, 2, 4, 8, 3, 6, 12, 24\}$. But if $n = 6$ or $n = 24$ the equality (10) can not be true because its left hand side has reminder 2 modulo 3, and the right hand side has reminder 1.

2nd case. $6\cdot 4^{p-1}/n$ is not divisible by 16. The number $6\cdot 4^{p-1}/n$ is a divisor of the number $6\cdot 4^{p-1}$, so it is equal to 2^m or $3\cdot 2^m$, and $6\cdot 4^{p-1}/n \leqslant 24$ as in the first case. Hence $n > 6\cdot 4^{p-1}/24 > 6\cdot 2^{p-1}$ (since $p \geqslant 8$). Then

$$6\cdot 4^{p-1} = 6\cdot 2^{p-1}(6\cdot 2^{p-1} - 5\cdot 2^{p-1}) < n(n - 5\cdot 2^{p-1}) < n\bigl(n - (3\cdot 2^{p-1} + 3)\bigr) = \frac{6\cdot 4^{p-1}}{n} - (2 + 3\cdot 2^{p-1} + 2\cdot 3^p) < 6\cdot 4^{p-1}\,.$$

A contradiction.

Much more bored considerations (with the help of computer) in [5] prove that the equation

$$n^4 - (2^{p+1} + 6)n^3 + (4 \cdot 3^p + 3 \cdot 2^{p+1} + 11)n^2 - (4 \cdot 3^p + 3 \cdot 2^{2p+2} + 6)n + 24 \cdot 5^{p-1} = 0$$

has no solutions in natural numbers and hence a set is determined by its set of 5-sums.

2.6. The answer is no. Such four sets do exist [8]. For example, $X = \{-8, -3, 1, 2, 3, 5\}$, $Y = \{-7, -3, 0, 1, 2, 7\}$, $-X$ и $-Y$.

2.7. [8] Extremely technical. Similarly to 1.6.

2.8. We take this statement from [12]. Use the formula from the problem 3.2. Multiply both sides by $k!$

$$k! f(n,k) = \sum_{i=1}^{k} (-1)^{i-1} i^{p-1} k! C_n^{k-i}.$$

Note that if $i \neq k$ then the corresponding term in the right hand side is divisible by n, and if $i = k$ the term is equal in absolute value to $k^{p-1}k!$. Hence the expression $f(n,k)$ equals zero only if $k^{p-1}k!$ is divisible by n. And otherwise: if $k^{n-1}k!$ is not divisible by n, then $f(n,k) \neq 0$ for $p = 1, 2, \ldots, n$. So if $k^{n-1}k!$ is not divisible by n, then due to the statement of the problem 3.2 and similarly to the problem 1.5 the values of the symmetric polynomials $s_p(A)$ are uniquely determined by the values of the polynomials $s_p(A^{(k)})$, for $p = 1, 2, \ldots, n$, and therefore the set A is uniquely determined.

2.9. Fomin and Izhboldin [3], theorem 5.1, prove this statement for the set B with linearly independent over $\mathbb{Q}$ elements. But they really need for the proof (that we cite below) the weaker property of the set B from our statement of the problem.

For a set of numbers X let Σ_X be the sum of elements of X.

Fix two numbers $x, y \in A$. There exist C_{n-2}^{k-1} unordered pairs of sets (X, Y) such that $X, Y \subset A$, these sets X and Y consist of k elements each and their symmetric difference $X \triangle Y = \{x, y\}$. And for any such pair $x - y = \Sigma_X - \Sigma_Y = \Sigma_{X \setminus Y} - \Sigma_{Y \setminus X}$.

Remark, by the way, that for any two subsets $X, Y \in B$ we can uniquely determine both sets $X \setminus Y$ and $Y \setminus X$ whenever we know their difference $\Sigma_X - \Sigma_Y$

For every pair (X, Y) we have the corresponding subsets X', Y' in the set B that have the same sums of elements i.e. $x - y = \Sigma_X - \Sigma_Y = \Sigma_{X'} - \Sigma_{Y'} = \Sigma_{X' \setminus Y'} - \Sigma_{Y' \setminus X'}$. Since the numbers x and y are fixed, the sets $S = X' \setminus Y' \subset B$ and $S = Y' \setminus X' \subset B$ do not really depend on the choice of the pair (X, Y). Hence for any pair of k-element subsets $U \subset A$, $V \subset A$ with the differences $U \setminus V = \{x\}$, $V \setminus U = \{y\}$ (there exist C_{n-2}^{k-1} such pairs) we have $U' \setminus V' = S$, $V' \setminus U' = T$. Assume that the set S consists of p elements. Then one can construct C_{n-2p}^{k-p} pairs of subset of the set B with the fixed differences of subsets. Since $C_{n-2p}^{k-p} < C_{n-2}^{k-1}$ for $p > 1$ then $p = 1$.

So we have obtained the correspondance that maps every (ordered) pair of numbers (x, y) from the set A into the pair (p, q) from B such that $x - y = p - q$. It is one-to-one correspondence due to the properties of the set B. Now take 3 pairs (x, y), (y, z) and (x, z), let they correspond to (p, q), (r, s), (u, v). Then $x - y = p - q$, $y - z = r - s$, $x - z = u - v$, hence $u - v = p + r - q - s$. So, the pairs (p, q) and (r, s) have nonempty intersection.

Consider now all the pairs (x, y_i), $i = 1, 2, \ldots, n-1$, and the corresponding pairs in the set B. Consider the case $n > 4$ only. (The case $n = 4$ is an easy problem that can be solved similarly to the problem 1.1.) The following "Helly theorem" is evident for $n > 4$: *if $n - 1$ pairs are chosen in the set of n elements such that any two pairs have nonempty intersection then all the pairs have a common point.* Hence all our pairs from the set B have a common element $\bar{x}$. It is clear that in all these ordered pairs it has the same position.

So all our pairs in B are of the form а) $(\bar{x}, y_i')$ or б) $(y_i', \bar{x})$.

In the case а) $\sum(x - x_i) = \sum(\bar{x} - y_i')$, hence $nx - \Sigma_A = n\bar{x} - \Sigma_B$, therefore $x = \bar{x}$ and so $y_i = y_i'$, i.e. $A = B$.

In the case б) $\sum(x - x_i) = \sum(y_i' - \bar{x})$, hence $nx - \Sigma_A = \Sigma_B - n\bar{x}$, therefore $x + \bar{x} = \frac{2\Sigma}{n}$, where $\Sigma = \Sigma_A = \Sigma_B$ и $y_i + y_i' = x + \bar{x} = \frac{2\Sigma}{n}$, i.e. the sets A and B are symmetric to each other with respect to the point Σ/n. If $n > k$, but $n \neq 2k$, then consider an arbitrary k-element set $X \subset A$. Let $\bar{X} \subset B$ be the set symmetric to X with respect to the point Σ/n, then $\Sigma_{\bar{X}} = \frac{2k\Sigma}{n} - \Sigma_X$. Let $X' \subset B$ be the set dual to X. Note that

$$\Sigma_{B \setminus \bar{X}} - \Sigma_{X'} = \frac{n - 2k}{n}\Sigma, \tag{11}$$

and after collecting the подобных terms all the terms in the left hand side have only ± 1 as coefficients. By the properties of the set B, the expression in the left hand side does not depend on the choice of the set X. To conclude the proof let us prove that it is possible only in the case $2k = n$ (and $X' = B \setminus \bar{X}$). Indeed, if the sum in the left hand side of (11) contains at least one nonzero term, say, $-b$, then consider an arbitrary k-element subset $Y' \subset B$, that does not contain number b. If we take in the equality (11) the set dual to Y' as a set X then the left hand side of the equality does not contain the term $-b$. Analogously, if the sum in the left hand side of (11) contains a nonzero term with a coefficient $+1$, say, $+b$, then we consider as a set X an arbitrary set containing the point $\frac{2\Sigma}{n}-b$.

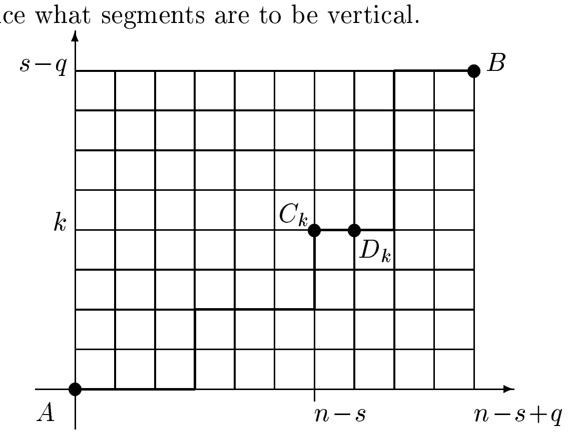
3 Addition after the semifinal

3.1. a) This subject is discussed in [1], chapter 6. There exist $(r-m)^k$ ways to put photographs in the envelopes in such a manner that m given envelopes would be empty. We can choose m envelopes of r by C_r^m ways. Now the formula is obtained by the standard inclusion-exclusion method.

б) This subject is discussed in [1], chapter 5. Let $k=s-v$ be a new parameter of the sum then we have to prove that

$$C_n^{s-q}=\sum_{v=q}^{s} C_{n-v}^{n-s}C_{v-1}^{q-1}=\sum_{k=0}^{s-q} C_{n-s+k}^{n-s}C_{s-k+1}^{q-1}\,. \tag{12}$$

Consider paths from the point $A(0,0)$ to the point $B(n-s+q,s-q)$ that go along the lines upwards or rightwards. The number of these paths is equal to C_n^{s-q} because every path consists of n segments and is determined by the choice what segments are to be vertical.



Let us count the number of these paths by another method. Draw the vertical line $x=n-s$ and mark in each path the highest point belonging to that line. For every point $C_k(n-s,k)$ consider all the paths which contain this marked point. To construct such a path we have to choose an arbitrary path from A to C_k (there exist C_{n-s+k}^{n-s} such paths) then walk along the horizontal segment C_kD_k and then continue by an arbitrary path from $D_k(n-s+1,k)$ to B (C_{s-k-1}^{q-1} ways). So, there exist $C_{n-s+k}^{n-s}C_{s-k-1}^{q-1}$ ways that contain a point C_k marked. Summing up by k we obtain exactly the right hand side of (12).

в) [8] Starting from the evident equality

$$s_{\alpha_1,\alpha_2,\ldots,\alpha_m}=s_{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}}\cdot s_{\alpha_m}-s_{\alpha_1+\alpha_m,\alpha_2,\ldots,\alpha_{m-1}}-s_{\alpha_1,\alpha_2+\alpha_m,\ldots,\alpha_{m-1}}-s_{\alpha_1,\alpha_2,\ldots,\alpha_{m-1}+\alpha_m}$$

we get by induction that for any positive integers α_1, α_2, ..., α_m the coefficient of $s_{\alpha_1+\alpha_2+\cdots+\alpha_m}$ in the expression $s_{\alpha_1,\alpha_2,\ldots,\alpha_m}$ via the symmetric functions s_i equals $(-1)^m(m-1)!$.

Note that if the coefficient p_k is allowed to have value 0, then

$$s_{p_1,\ldots,p_{k-1},0}=(n-k+1)s_{p_1,\ldots,p_{k-1}}\,. \tag{13}$$

Now if exactly t numbers of p_1, ..., p_k are equal to 0 then we can assume without loss of generality that $\{p_1,\ldots,p_k\}=\{\ell_1,\ldots,\ell_{k-t},0,\ldots,0\}$, where all ℓ_i are positive. Applying consequently (13), we find that

$$s_{p_1,\ldots,p_k}=(n-k+1)(n-k+2)\ldots(n-k+t)s_{\ell_1,\ldots,\ell_{k-t}}$$

To conclude the proof use the statement from the first paragraph of the solution.

3.2. [8].

$$k!s_p(A^{(k)})=\sum_{*}(a_{i_1}+\ldots+a_{i_k})^p=\sum_{*}\sum_{\diamond}\frac{p!}{p_1!\ldots p_k!}\,a_{i_1}^{p_1}\ldots a_{i_k}^{p_k}=$$
$$=\sum_{\diamond}\frac{p!}{p_1!\ldots p_k!}\sum_{*}a_{i_1}^{p_1}\ldots a_{i_k}^{p_k}=\sum_{\diamond}\frac{p!}{p_1!\ldots p_k!}\,s_{p_1,\ldots,p_k}(A)\,, \tag{14}$$

where the sum marked with star denotes the sum over all ordered sets $\{i_1, i_2, \ldots, i_k\} \subset A$; and the sum marked with rhombus denotes the sum over all ordered sets of nonnegative integers $(p_1, \ldots, p_k)$, $\sum p_i = p$. We know from the problem 3.1в) that the coefficient K of $s_p(A)$ in the rightmost hand side of (14) equals

$$K = \sum_{t=0}^{k-1} \frac{(n-k+t)!}{(n-k)!} \cdot (-1)^{k-t-1} \cdot (k-t-1)! \sum_{\square} \frac{p!}{p_1! \ldots p_k!},$$

where the sum marked with square denotes the sum over all ordered sets of nonnegative integers $(p_1, \ldots, p_k)$, $\sum p_i = p$, such that exactly t numbers p_i are equal to 0. By the result of the problem 3.1a) we have

$$\sum_{\square} \frac{p!}{p_1! \ldots p_k!} = C_k^t \sum_{\ell=1}^{k-t} (-1)^{k-t-\ell} C_{k-t}^{\ell} \ell^k.$$

Substituting of this expression in the previous equality gives us

$$\begin{aligned}
K &= \sum_{t=0}^{k-1} \frac{(n-k+t)!}{(n-k)!} \cdot (-1)^{k-t-1} \cdot (k-t-1)! C_k^t \sum_{\ell=1}^{k-t} (-1)^{k-t-\ell} C_{k-t}^{\ell} \ell^k = \\
&\underset{v:=k-t}{=} \sum_{v=1}^{k} \frac{(n-v)!}{(n-k)!} \cdot (-1)^{v-1} \cdot (v-1)! \frac{k!}{v!(k-v)!} \sum_{\ell=1}^{k-t} (-1)^{v-\ell} \frac{v!}{\ell!(v-\ell!)} \ell^k = \\
&= k! \sum_{\ell=1}^{k} (-1)^{\ell-1} \ell^{k-1} \sum_{v=\ell}^{k} C_{n-v}^{k-v} C_{v-1}^{\ell-1} = \\
&\underset{3.16}{=} k! \sum_{\ell=1}^{k} (-1)^{\ell-1} \ell^{k-1} C_n^{k-\ell}.
\end{aligned}$$

3.3. If $p = 1$,

$$f(n,k) = C_n^{k-1} - C_n^{k-2} + C_n^{k-3} - \ldots (-1)^{k-1} C_n^0.$$

This expression is a polynomial in n if we rewrite the binomial coefficients as polynomials in n as follows $C_n^m = \frac{n(n-1)(n-2)\ldots(n-m+1)}{m!}$. It is evident that $n = 1$, $n = 2$, $\ldots$, $n = k-1$ are the roots of this polynomial. For example when $n = 4$ (and $k - 1 \geqslant 4$) the expressions C_n^{k-1}, $\ldots$, C_n^5 are equal to 0 and

$$f(4,k) = \pm(C_4^4 - C_4^3 + C_4^2 - C_4^0 + C_4^0) = 0. \tag{15}$$

So, $f(n,k)$ is a polynomial of degree $k-1$ with respect to n and we know its $k-1$ roots. Hence $f(n,k) = \frac{1}{(k-1)!}(n-1)(n-2)\ldots(n-k+1) = C_{n-1}^{k-1}$. The constant is chosen to make leading coefficients in both sides equal.

3.4. We take this question in [3] and the answer in [12]. For $p = 1$ we have used in the previous problem the equality (15) that is obtained from the relation

$$(1-x)^m = C_m^m - C_m^{m-1} x + C_m^{m-2} - \ldots + (-1)^m C_m^0 x^m \tag{16}$$

by the substitution $x = 1$ (and $m = 4$). Taking a derivative of (16) with respect to x we obtain

$$m(1-x)^{m-1} = C_m^{m-1} - 2C_m^{m-2} x + 3C_m^{m-3} x^2 \ldots + (-1)^{m-1} m C_m^0 x^{m-1}. \tag{17}$$

By the substitution $x = 1$ we obtain that for $m > 1$

$$C_m^{m-1} - 2C_m^{m-2} x + 3C_m^{m-3} x^2 \ldots + (-1)^{m-1} m C_m^0 x^{m-1} = 0.$$

Due to this equality and similar to the previous problem we discover that for $p = 2$

$$f(n,k) = \frac{1}{(k-1)!}(n-2)(n-3)\ldots(n-k) = C_{n-2}^{k-1}.$$

This means that for $p = 2$ the equation $f(n,k) = 0$ has no interesting solutions.

Analogously for $p = 3$ one can obtain that $(k-1)! f(n,k) = (n-2k) \prod_{i=3}^{k} (n-i)$. Here we have a bit more interesting root $n = 2k$ here, whose existence we have observed in the problem 2.3a).

Consider the case $p=4$. Multiply both sides of (17) by x, take a derivative and repeat these operations one more time, we will obtain

$$C_m^{m-1}-2^3C_m^{m-2}x+3^3C_m^{m-3}x^2+\ldots+(-1)^{m-1}m^3C_m^0x^{m-1}=\frac{d}{dx}\Bigl(x\frac{d}{dx}\Bigl(xm(1-x)^{m-1}\Bigr)\Bigr)=(1-x)^{m-3}g(x)\,,$$

where $g(x)$ is a some quadratic polynomial of x (with coefficients depending on m). By substituting $x=1$ we obtain that for $n>3$

$$C_m^{m-1}-2^3C_m^{m-2}+3^3C_m^{m-3}+\ldots+(-1)^{m-1}m^3C_m^0=0\,.$$

Using this equality we found that for $p=4$ the expression $(k-1)!f(n,k)$ as a polynomial of n has roots $n=4$, $n=5$, ... , $n=k$. Hence

$$(k-1)!f(n,k)=(k-1)!(C_n^{k-1}-2^3C_n^{k-2}+3^3C_n^{k-3}-\ldots(-1)^{k-1}k^3C_n^0)=h(n)\prod_{i=4}^{k}(n-i)\,, \tag{18}$$

where $h(n)=n^2+an+b$ is a quadratic polynomial. By the substitution $n=0$ in (18) we found that $b=6k^2$, by the substitution $n=1$ we obtain $a=1-6k$. Then

$$h(n)=n^2-(6k-1)n+6k^2=\frac{1}{4}\bigl((2n-6k+1)^2-3(2k-1)^2+2\bigr)\,.$$

So, nontrivial roots of the equation $f(n,k)=0$ are the roots of Pell equation

$$(2n-6k+1)^2-3(2k-1)^2=-2\,. \tag{19}$$

The general solution of the Pell equation $u^2-3v^2=-2$ is of the form $u+v\sqrt{3}=\pm(1+\sqrt{3})(2+\sqrt{3})^r$, $r=0$, ±1, ±2, ... But we can use more simple idea that gives by the way all positive solutions. From the problem 1.5 we know that for $p=4$ a pair $(n,k)=(8,2)$ is a solution. Then use that the equation (19) has degree two both on n and s. Making the substitution $n=8$ in (19) we found one more solution $(8,6)$. By the way, we already know that the set of 8 numbers is not determined by its 6-sums from the problem 2.3б). Making the substitution $k=6$ in (19) we found one more solution $(27,6)$. The substitution $n=27$ gives one more solution $(27,21)$. The substitution $k=21$ gives $n=98$ etc. We discover the sequence of solution

$$(8,2),\quad (8,6),\quad (27,6),\quad (27,21),\quad (98,21),\quad (98,77),\quad (363,77)\quad \ldots$$

As they mention in [12], for $p=5$ $f(n,k)=(n^2-(12n-5)k+12k^2)(n-2k)\prod_{i=5}^{k}(n-i)$ and the corresponding Pell equation $u^2-6v^2=75$ gives solutions $(16,2)$, $(147,14)$, $(1444,133)$, ... or $(27,3)$, $(256,24)$, $(2523,232)$ ... .

Remark that for $p=6$ the search for nontrivial solutions of the equation $f(n,k)=0$ leads to the solving the equation of 4th degree. This equation is a subject of the next problem.

3.5. No ideas. Searching by computer shows that all the solutions in the square $|n|<2000$, $|m|<2000$ are listed in the statement of the problem.

3.6. [12]. For $k=2$ and all p we have

$$s_p(A^{[2]})=\sum_{1\leqslant i_1\leqslant i_2\leqslant n}(a_{i_1}+a_{i_2})^p=\frac{1}{2}\biggl(\sum_{i_1,i_2=1}^{n}(a_{i_1}+a_{i_2})^p+\sum_{i=1}^{n}(2a_i)^p\biggr)\,.$$

For $k=3$

$$s_p(A^{[3]})=\sum_{1\leqslant i_1\leqslant i_2\leqslant i_3\leqslant n}(a_{i_1}+a_{i_2}+a_{i_3})^p=\frac{1}{6}\biggl(\sum_{i_1,i_2,i_3=1}^{n}(a_{i_1}+a_{i_2}+a_{i_3})^p+3\sum_{i_1,i_2=1}^{n}(2a_{i_1}+a_{i_2})^p+2\sum_{i=1}^{n}(3a_i)^p\biggr)\,.$$

And so on for other k. Rewriting the expression similarly to the problem 2.2 we will obtain that the right hand sides of these equalities have a form $q(n,k)s_k(A)+G$, where G is an expression that depends on n and $s_i(A)$, $i<k$, and $q(n,k)>0$. Then as in the problem 1.5 the power sums of the set A is determined by the power sums of the set $A^{[k]}$. Hence the set A is determined by the set $A^{[3]}$.

References

[1] Виленкин Н.Я. Комбинаторика. М., 1969.

[2] Гальперин Г.А., Толпыго А.К. Московские математические олимпиады. М., 1986.

[3] Фомин Д.В., Ижболдин О.Т. Наборы кратных сумм. Труды Санкт-Петербургского математического общества. Т. 3. С. 244–259.

[4] Фомин Д.В. Санкт-Петербургские математические олимпиады. СПб. 1994

[5] Amdeberhan T., Zeleke M. On injectivity of combinatorial Radon transform of order five. www.math.temple.edu/˜tewodros

[6] Boman J., Linusson S. Examples of non-uniqueness for the combinatorial Radon transform modulo the symmetric group. Math. Scand. V. 78 (1996). No. 2. 207–212.

[7] Erdős P. Some of my favorite unsolved problems. Math. Japonica V. 46. No. 3 (1997). P. 527–538.

[8] Ewell J.A. On the determination of sets by sets of sums of fixed order. Canad. J. Math., V. 20 (1968). P. 596–611.

[9] Gordon B., Fraenkel A.S., Straus E.G. On the determination of sets by the sets of sums of a certain order. Pacific J. Math. V. 12 (1962), P. 187–196.

[10] Guy R. Unsolved problems in number theory.

[11] Moser L. Amer. Math. Monthly. V. 64 (1957). No. 7. P. 507.

[12] Selfridge J.L., Straus E.G. On the determination of numbers of sets by the sets of their sums of a fixed order. Pacific J. Math. V. 8 (1958), P. 847–856.