Catalan Numbers and Natural Mappings

Catalan numbers often occur in mathematics as cardinalities[1] of different sets (usually consisting of combinatorial objects). As a result one can formulate lots of theorems like: "The cardinality of the set A is equal to the cardinality of the set B". As these theorems hold, it would be interesting to prove them constructing natural (in a reasonable sense) mappings — each being a one-to-one correspondence[2] — from A to B. I know several mappings involving some of the sets described below; for almost all pairs of these sets it is possible to construct a mapping between the sets of this pair using composition of mappings. Unfortunately, some mappings become very complicated. I hope that some new mappings will be constructed by the participants of the Conference. To make any undefined terminology clear, each problem is provided with an example of the corresponding set for $n = 3$. The ordering of the elements in examples is random and has no special meaning.

Definition. *Catalan numbers c_n ($n \geqslant 0$) are defined by the recurrence relation $c_{n+1} = c_0c_n + c_1c_{n-1} + \cdots + c_nc_0$ ($n \geqslant 0$) and the "initial condition" $c_0 = 1$.*

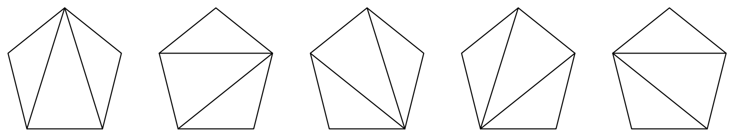
These problems are **not** about the explicit formula for Catalan numbers. But ...

1. (The explicit formula) Prove that $c_n = \dfrac{(2n)!}{n!(n+1)!}$.

The next problem contains several combinatorial interpretations.

2. Prove (bijectively if possible) that all these sets consist of c_n elements.

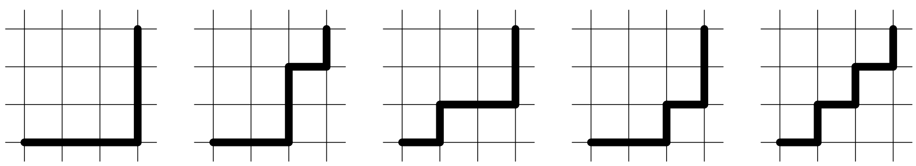
(a) *Triangulations* of a convex $(n+2)$–gon into n triangles (by $n-1$ diagonals)



(b) Non-associative products of $n+1$ variables (being less formal, possible orders of performing multiplications)

`a(b(cd))`   `(ab)(cd)`   `((ab)c)d`   `a((bc)d)`   `(a(bc))d`

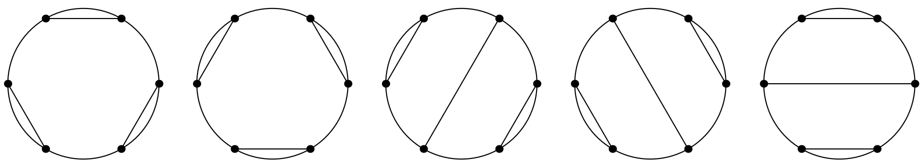
(c_1) Lattice paths (with steps $(0,1)$ and $(1,0)$) from $(0,0)$ to (n,n), never rising above the diagonal $y = x$



(c_2) Sequences of 1's and -1's such that 1 occurs n times, -1 ocuurs n times, all partial sums[3] are non-negative

`1 1 1 -1 -1 -1`   `1 1 -1 1 -1 -1`   `1 -1 1 1 -1 -1`   `1 -1 1 -1 1 -1`   `1 1 -1 -1 1 -1`

(d) Ways of connecting $2n$ points on a circle with n nonintersecting chords



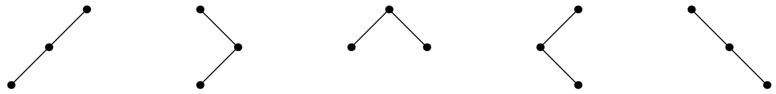
[1] Cardinality of a set is just the number of elements in this set

[2] Sometimes people use a slightly less clear word "bijection" instead of "one-to-one correspondence"

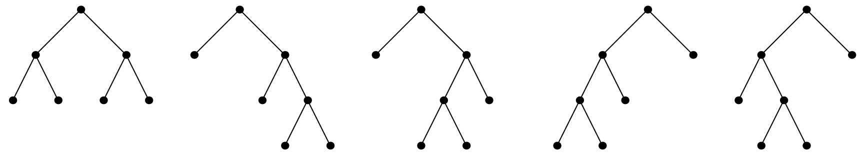
[3] By kth partial sum of the sequence $a_1, \ldots, a_p$ we mean $a_1 + \cdots + a_k$

The next problem suggests more interpretations.
3. Prove (bijectively if possible) that all these sets consist of c_n elements.

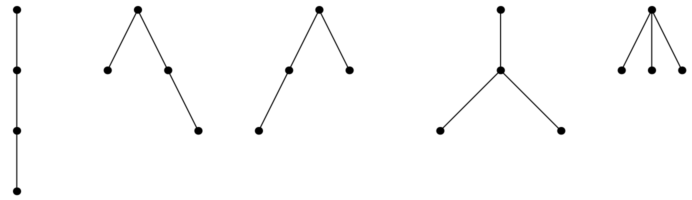
(a) *Flat rooted binary trees* (each vertex has at most two sons ["left and/or right"] and one parent [except for the root which (by definition) has no parents]) with n vertices



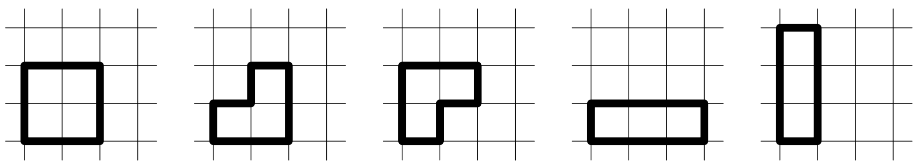
(b) *Flat rooted strictly binary trees* with $n+1$ endpoints ("leaves"): each vertex has either two sons or none



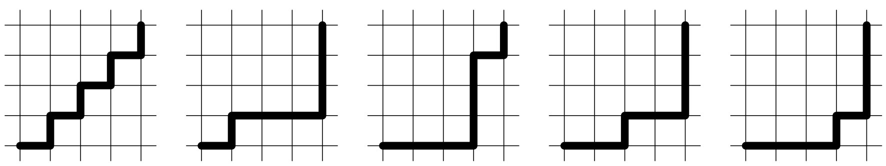
(c) Flat rooted trees with $n+1$ vertices



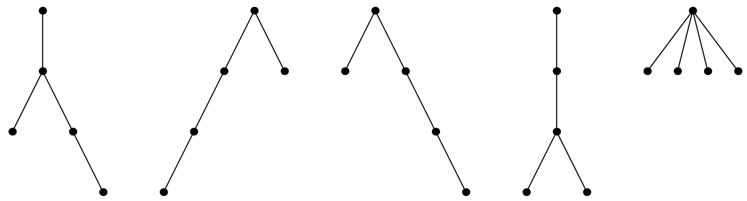
(d) Parallelogram polyominoes (unordered pairs of lattice paths (with steps $(0,1)$ and $(1,0)$) starting at $(0,0)$, ending at the same point and only intersecting at the beginning and end) of perimeter $2n+2$



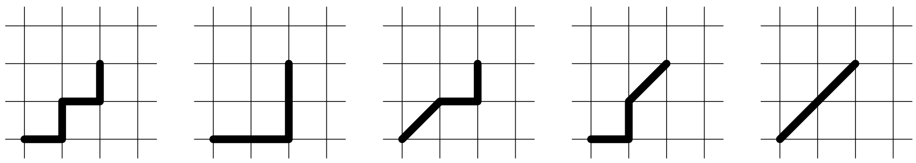
(e_1) Lattice paths as in 2(c_1) from $(0,0)$ to $(n+1, n+1)$ such that any vertical side ending on the diagonal has odd length



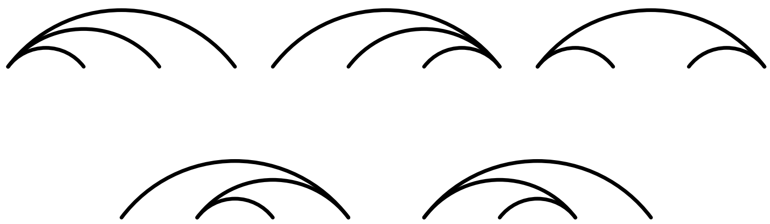
(e_2) Flat rooted trees with $n+2$ vertices such that the rightmost path of each subtree of the root has even length



(f) Lattice paths (with steps $(0,1)$, $(1,0)$ and $(1,1)$) from $(0,0)$ to $(n-1,n-1)$, never rising above the diagonal $y=x$ and such that steps $(1,1)$ are permitted only along $y=x$



(g) Ways of connecting $n+1$ points on a line with n noncrossing arcs such that the graph thus formed has no cycles and at every vertex all the arcs exit in the same direction (left or right)



(h) Integer sequences $1 \leqslant a_1 \leqslant \ldots \leqslant a_n$ with $a_i \leqslant i$

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1 1 1     1 1 2     1 1 3     1 2 2     1 2 3
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(i) Integer sequences $a_1, \ldots, a_{n-1}$ such that $a_i \leqslant 1$ and all partial sums are nonnegative

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0 0     0 1     1 0     1 1     1 -1
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(i_1) Integer sequences $a_1, \ldots, a_n$ such that $a_i \geqslant -1$, all partial sums are nonnegative, and $a_1+a_2+\cdots+a_n = 0$

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0 0 0     0 1 -1     1 0 -1     1 -1 0     2 -1 -1
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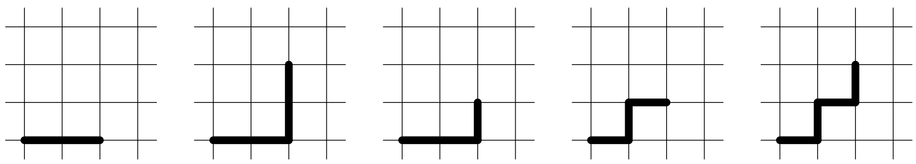
(j) Lattice paths (with steps $(0,1)$ and $(1,0)$) starting at $(0,0)$, never rising above $y=x$ such that the number of horizontal steps is $n-1$



(k) Integer sequences $a_1, \ldots, a_n$ such that $a_1 = 0$ and $0 \leqslant a_{i+1} \leqslant a_i + 1$

0 0 0 0 0 1 0 1 0 0 1 1 0 1 2

(l) Integer sequences $a_1, \ldots, a_{2n}$ such that $a_i \geqslant 0$, $a_1 = 1$, $a_{2n} = 0$, and $a_i - a_{i+1} = \pm 1$

1 2 3 2 1 0 1 2 1 2 1 0 1 2 1 0 1 0 1 0 1 2 1 0 1 0 1 0 1 0

(m) Permutations of $1, 2, \ldots, n$ with longest decreasing subsequence of length at most two

1 2 3 2 1 3 1 3 2 3 1 2 2 3 1

(n) Permutations of $1, 2, \ldots, 2n$ such that even numbers appear in increasing order, odd numbers appear in decreasing order and each odd number appears before the consecutive even number ("$2k-1$ appears before $2k$")

1 2 3 4 5 6 1 2 3 5 4 6 1 3 2 4 5 6 1 3 2 5 4 6 1 3 5 2 4 6

(o) *Noncrossing* partitions of $1, \ldots, n$ into several blocks (for $a < b < c < d$ if a and c belong to the same block and b and d belong to the same block then a, b, c, d belong to the same block)

123 12-3 13-2 23-1 1-2-3

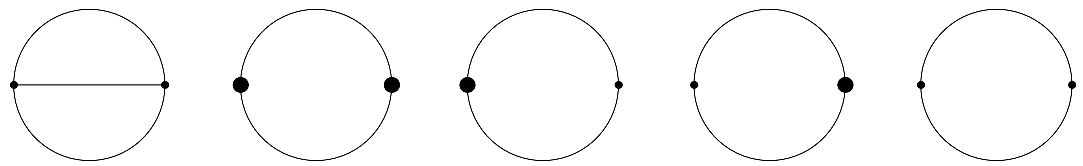
(p) *Noncrossing* (see (o)) partitions of $1, \ldots, 2n+1$ into $n+1$ blocks such that no block contains two consecutive integers

137-46-2-5 1357-2-4-6 157-24-3-6 17-246-3-5 17-26-35-4

(q) Tables with 2 rows and n columns filled with numbers $1, \ldots, 2n$ such that all rows and columns are increasing

1	2	3	1	2	4	1	3	4	1	2	5	1	3	5
4	5	6	3	5	6	2	5	6	3	4	6	2	4	6

(r) Ways of connecting some of $n-1$ points on a circle with nonintersecting chords and marking some of the remaining points



(s) Unordered n–tuples of (not necessarily different) integers from $0, 1, \ldots, n$ such that the sum of all numbers is divisible by $n+1$

0 0 0 0 1 3 0 2 2 1 1 2 2 3 3

(t) Integer sequences $a_1, \ldots, a_n$ ($a_k \geqslant 2$) such that in the sequence $1, a_1, \ldots, a_n, 1$ each a_i divides the sum of its two neighbours

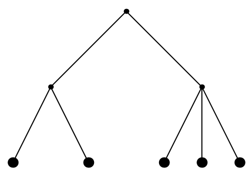
4 3 2 3 5 2 3 2 3 2 5 3 2 3 4

(u) Integer sequences $a_1, \ldots, a_n$ such that a_i is equal to the number of $j < i$ such that $a_j < a_i$

0 0 0   0 0 2   0 1 0   0 1 1   0 1 2

(v) Vertices at level $n-1$ of the tree defined as follows:

- The root (the unique vertex of the zero level) has two sons
- If v (at level q) has k sons, its sons (at level $q+1$) have $2, 3, \ldots, k+1$ sons



4. Prove (bijectively if possible) that the number of lattice paths (with steps $(0,1)$ and $(1,0)$) from $(0,0)$ to $(2n,2n)$ that avoid all the points of type $(2k-1, 2k-1)$ is equal to c_{2n}.