Catalan Numbers and Natural Mappings: After the Semifinal

3.

(w) Integer sequences $a_1, \dots, a_n$ with $0 \leqslant a_i \leqslant n - i$ such that if $i < j, a_i > 0, a_j > 0$ and $a_{i+1} = \dots = a_{j-1} = 0$, then $a_i + i < a_j + j$

0 0 0   0 1 0   1 0 0   2 0 0   1 1 0

(x) Integer sequences $a_1, \dots, a_n$ such that $1 \leqslant a_i \leqslant i$ and if $a_i = j$, then $a_{i-p} \leqslant j - p$ for $1 \leqslant p \leqslant j - 1$

1 1 1   1 1 2   1 1 3   1 2 1   1 2 3

(y) Permutations $a_1, \dots, a_n$ of $1, 2, \dots, n$ for which there does not exist $i < j < k$ with $a_j < a_k < a_i$ ("312-avoiding" permutations)

1 2 3   1 3 2   2 1 3   2 3 1   3 2 1

Definition. *A standard Young n-tableaux is a table with nonincreasing lengths of rows filled with numbers $1, \dots, n$ (each number occurs once) such that all rows and columns are increasing.*

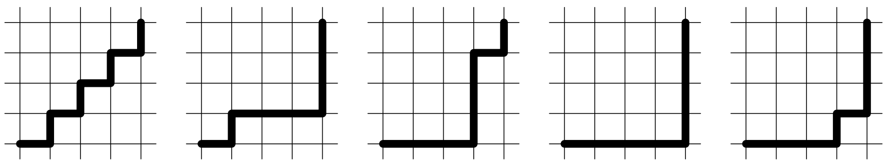
Examples of standard Young 6-tableaux:

1	2	3		1	2	4	5		1	2
4	5			3	6				3	5
6									4	6

(z) Pairs (P, Q) of standard Young n-tableaux of the same shape with at most two rows

1 2 3	1 2 3	1 2	1 2	1 2	1 3	1 3	1 2	1 3	1 3
		3	3	3	2	2	3	2	2

($\hbar$) Lattice paths as in 2(c_1) from $(0, 0)$ to $(n + 1, n + 1)$, no horizontal side ending on the line $y = x - 2$.



5. Prove that if there exists a partition of $1, 2, \dots, k$ into $n + 1$ noncrossing blocks such that no block contains two consecutive integers,[1] then $k \leqslant 2n + 1$.

Definition. *Narayana numbers $N_{n,k}$ are defined by the formula*[2]

$$N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

6. Prove that the set of sequences from 2(c_2) with exactly k subwords '`1 -1`' consists of $N_{n,k}$ elements.

7. It follows from the previous problem that for each n

$$\frac{1}{n}\binom{n}{1}\binom{n}{0} + \frac{1}{n}\binom{n}{2}\binom{n}{1} + \dots + \frac{1}{n}\binom{n}{n}\binom{n}{n-1} = c_n.$$

Derive from the bijections constructed more identities involving $\binom{n}{k}, c_n, \dots$

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[1] As in 3(p)

[2] We use the standard notation $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$