Emelyanov L.A., Emelyanova T.L.

Three mysterious points of triangle

Introduction

Let ABC be a triangle. By I denote the center of the inscribed circle of ABC, by A', B', C' denote the touch points of the incircle and the sides BC, CA, AB, respectively. By L_1, L_2, L_3 denote the mirror images of I in the sidelines of the triangle $A'B'C'$ (fig. 1).

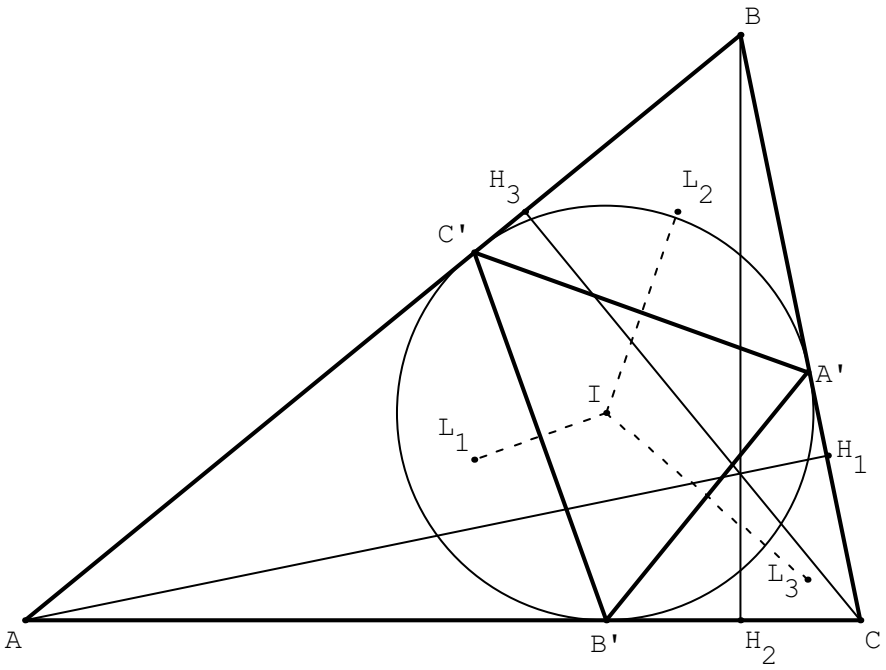

Fig. 1

Further we use the following notation. Let H_1, H_2, H_3 be the feet of the altitudes of ΔABC (Fig. 1); denote $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$; $AB = c$, $BC = a$, $AC = b$. By (XYZ) denote the circle passing through the points X, Y, Z. By O denote the circumcenter of ΔABC. The triangle $A'B'C'$ is called the *intouch* triangle.

We suggest to prove the following results concerning the points L_1, L_2, L_3.

1. Reflect the circumcenter of a triangle with respect to it's sidelines. The three obtained points are the vertices of a triangle that is symmetric to the original triangle with respect to a point.

Corollary. $\Delta L_1L_2L_3$ is symmetric to $\Delta A'B'C'$ with respect to a point.

2. L_1, L_2, L_3 are the ortocenters of the triangles $AB'C'$, $BC'A'$, $CA'B'$, respectively.

3. The bisectors of ΔABC are the altitudes of $\Delta L_1L_2L_3$. Thus I is the orthocenter of $\Delta L_1L_2L_3$. Similarly, the orthocenter H' of the intouch triangle is the center of $(L_1L_2L_3)$.

4. L_1, L_2, L_3 are the incenters of the triangles AH_2H_3, BH_3H_1, CH_1H_2, respectively.

5. The common pairwise extangents of the incircles of the triangles AH_2H_3, BH_3H_1 и CH_1H_2 (distinct from the sidelines of ΔABC) have a common point that is the center of $(L_1L_2L_3)$ (fig.2).

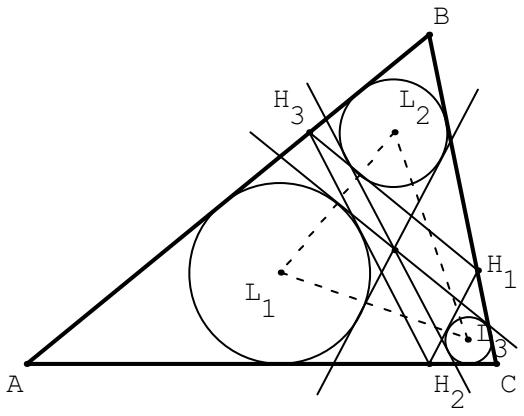

Fig. 2

6. The common extangents (see Problem 5) are parallel to the sidelines of the orthic triangle $H_1H_2H_3$.

7. The lines L_1H', L_2H', L_3H' (see Problem 3) are perpendicular to the sidelines of ΔABC. (*Definition.* Let XYZ and $X'Y'Z'$ be triangles. Suppose the lines passing through X, Y, Z perpendicular to $Y'Z'$, $Z'X'$, $X'Y'$, respectively, have a common point Q. Then the triangle XYZ is called *orthologic* to the triangle $X'Y'Z'$ with respect to Q.)

8. The points A, B, L_1, L_2 are concyclic; denote by ω_3 the circle passing through A, B, L_1, L_2. The similar result is true for quadruples B, C, L_2, L_3 (the circle ω_1) and A, C, L_1, L_3 (the circle ω_2).

9. The triangles $H_1H_2H_3$ and $L_1L_2L_3$ are perspective. (The triangles XYZ and $X'Y'Z'$ are called *perspective* if the lines XX', YY', ZZ' have a common point.)

10. Let A_0, B_0 и C_0 be the midpoints of the sides BC, CA, AB, respectively. The triangles $A_0B_0C_0$ and $L_1L_2L_3$ are perspective.