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Three mysterious points of triangle

Solution of Introduction

1. By K_0, M_0 denote the midpoints of the sides MN, KN of ΔKMN, by Q denote the circumcenter of ΔKMN . Let K_1 and M_1 be the points symmetrical to Q with respect to MN and KN (fig. 3).

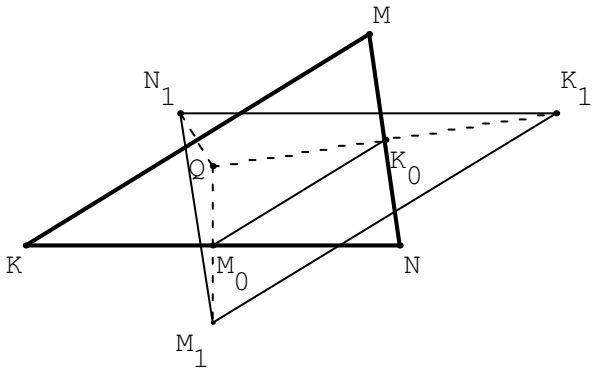

Fig. 3

The segment K_0M_0 is a common medial line of triangles KMN, K_1QM_1, hence the segments K_1M_1, KM are equal and parallel. The same is true for other pairs of sides. From this it follows that ΔKMN and $\Delta K_1M_1N_1$ are symmetrical with respect to a point.

2. Let us prove this statement for $\Delta AB'C'$ (fig. 4).

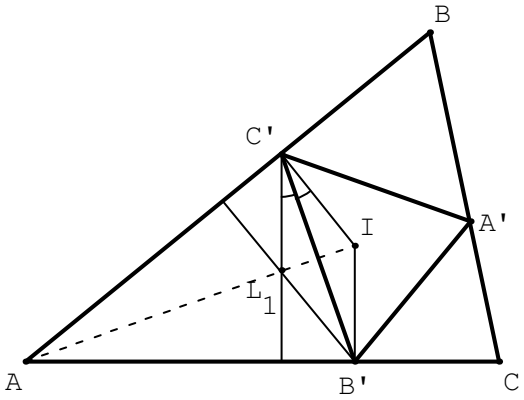

Fig. 4

$\angle L_1C'I = \angle IC'B' = \angle IB'C'$, hence $C'L_1 \parallel IB'$. From this it follows that $C'L_1 \perp AB'$. Similarly, $B'L_1 \perp AC'$. Hence, L_1 is the orthocenter of $\Delta AB'C'$.

3. The bisector AL_1 is perpendicular to $B'C'$ and L_2L_3 ($B'C' \parallel L_2L_3$). Hence AL_1 is the altitude in $\Delta L_1L_2L_3$. Similarly, BL_2 and CL_3 are the altitudes in $\Delta L_1L_2L_3$.

4. Let us prove the statement for L_1 (fig. 5).

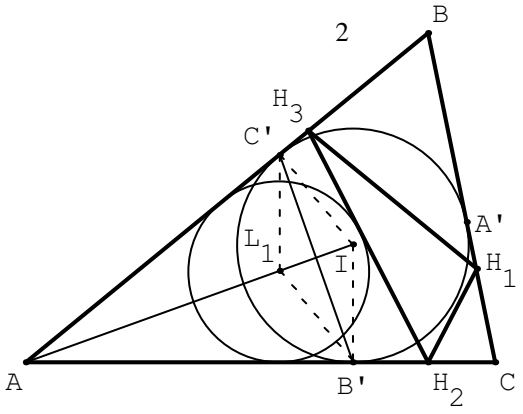

Fig. 5

The points B, C, H_2, H_3 lie on a circle with diameter BC. Hence, $\angle AH_3H_2 = \angle ACB$. This means that the triangles AH_2H_3 and ABC are similar with ratio $\frac{AH_3}{AC} = \cos\alpha$. From this it follows that the distance from A to the incenter of ΔAH_2H_3 equals $AI\cos\alpha$. From the other hand,

$$AL_1 = AI - IL_1 = AI - 2IB'\sin\frac{\alpha}{2} = AI - 2AI\sin^2\frac{\alpha}{2} = AI\cos\alpha.$$

Thus L_1 is the incenter of ΔAH_2H_3.

5. (fig. 6)

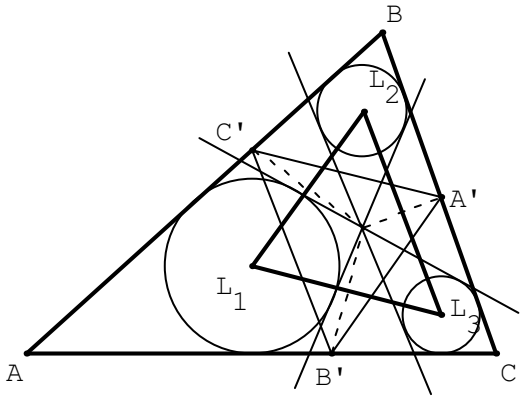

Fig. 6

The triangles $A'B'C'$ and $L_1L_2L_3$ are symmetrical with respect to a point. The points L_1, L_2, L_3 were obtained from the circumcenter of $\Delta A'B'C'$ by reflection with respect to it's sidelines. Therefore A', B', C' could be obtained from the circumcenter of $\Delta L_1L_2L_3$ by reflection with respect to it's sidelines (note that these sidelines are the centerlines for three pairs of circles). The points A', B', C' lie on the sidelines of ΔABC that is the pairwise extargents to the circles. Therefore the circumcenter of $\Delta L_1L_2L_3$ lies on the second pairwise extangents to the circles.

6. $\angle AH_3H_2 = \angle ACB$ (see solution of Problem 4), hence the reflection with respect to the bisector of the angle A takes H_2H_3 to a line parallel to BC. From the other hand, the second common extangent l to the incircles of triangles BH_3H_1, CH_1H_2 is symmetric to BC with respect to L_2L_3. Note that L_2L_3 is perpendicular to the bisector of A. Therefore the reflection with respect to the bisector of A takes l to a line parallel to BC. This means that l is parallel to H_2H_3. (fig. 7)

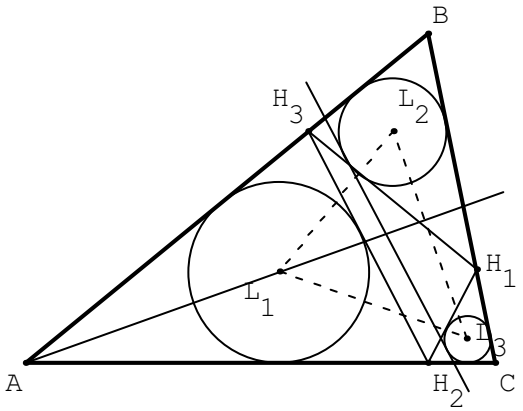

Fig. 7

7. By P and Q denote the common point of BC and the common extangents that are parallel to H_1H_2 and H_1H_3 (fig. 8). L_1H' is the bisector of the angle between these extangents.

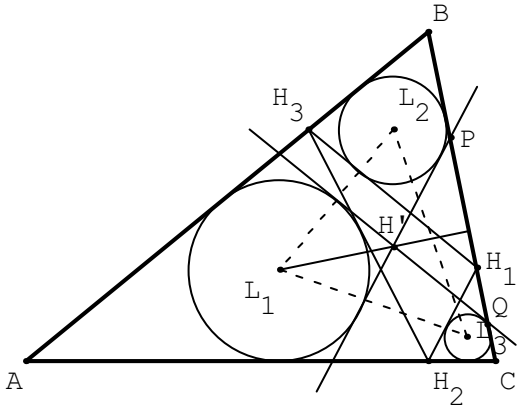

Fig. 8

$\angle H'PQ = \angle H_2H_1C = \angle H_3H_1B = \angle H'QP$, hence $\Delta H'PQ$ is equilateral. This implies that L_1H' is the altitude of $\Delta H'PQ$ that is $L_1H' \perp BC$.

8. Let us prove that B, C, L_2, L_3 are concyclic. Let T be the intersection point of lines AI and L_2L_3 (fig. 9).

$$\angle BL_2L_3 = \angle L_2IT + \angle L_2TI = \frac{\alpha}{2} + \frac{\beta}{2} + 90° = 180° - \frac{\gamma}{2}.$$

Therefore BCL_3L_2 is an inscribed quadrilateral.

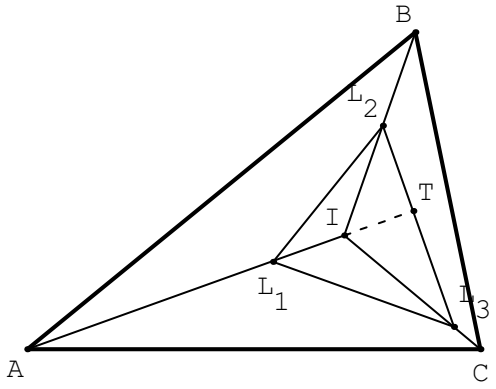

Fig. 9

9. Use Ceva theorem for the orthic triangle $H_1H_2H_3$. Find the ratio in which H_1L_1 divides the segment H_2H_3. By D denote the common point of H_1L_1 and H_2H_3, by E and F denote projections of D on H_1H_3 and H_1H_2, by S and G denote projections of L_1 on H_1H_3 and H_1H_2 (fig 10).

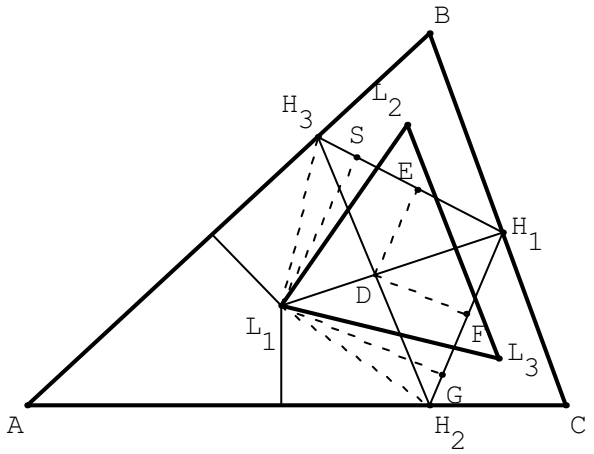

Fig. 10

Taking into account that $\angle H_1H_2H_3 = 180° - 2\beta$ and $\angle H_1H_3H_2 = 180° - 2\gamma$, we get

$$\frac{H_2D}{H_3D} = \frac{DF/\sin 2\beta}{DE/\sin 2\gamma} = \frac{L_1G\sin 2\gamma}{L_1S\sin 2\beta} = \frac{L_1H_2\sin\frac{3\beta}{2}\sin 2\gamma}{L_1H_3\sin\frac{3\gamma}{2}\sin 2\beta} = \frac{\sin\frac{\gamma}{2}\sin\frac{3\beta}{2}\sin 2\gamma}{\sin\frac{\beta}{2}\sin\frac{3\gamma}{2}\sin 2\beta}.$$

The ratio for the other sides of the orthic triangle equals

$$\frac{\sin\frac{\alpha}{2}\sin\frac{3\gamma}{2}\sin 2\alpha}{\sin\frac{\gamma}{2}\sin\frac{3\alpha}{2}\sin 2\gamma} \text{ and } \frac{\sin\frac{\beta}{2}\sin\frac{3\alpha}{2}\sin 2\beta}{\sin\frac{\alpha}{2}\sin\frac{3\beta}{2}\sin 2\alpha}, \text{ respectively.}$$

It can easily be checked that the product of these three fractions equals 1. This completes the proof.

10. Use Ceva theorem for the medial triangle $A_0B_0C_0$. Find the ratio in which B_0L_2 divides the segment C_0A_0 (by N denote the intersection point, see fig. 11).

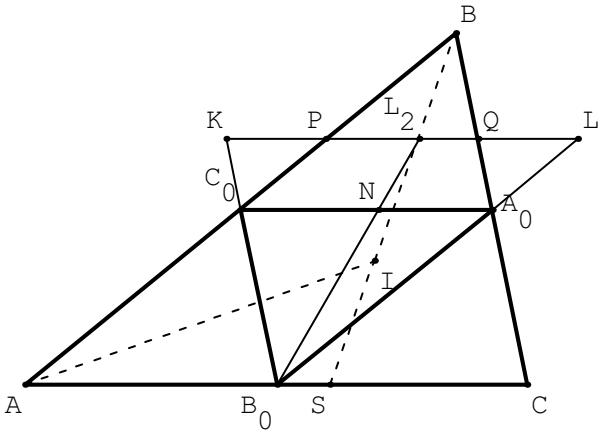

Fig. 11

Let K and L be the intersection point of the line m passing through L_2 parallel to AC with B_0C_0 and B_0A_0, respectively. By P and Q denote the intersection points of m with AB and BC,

respectively. Suppose the similarity ratio of triangles PBQ and ABC equals k, then PBQ and $A_0B_0C_0$ are similar with ratio $2k$, and each of triangles KC_0P, QA_0L is similar to $A_0B_0C_0$ with ratio $1-2k$. We have

$$\frac{C_0N}{A_0N}=\frac{KL_2}{LL_2}=\frac{KP+PL_2}{LQ+QL_2}=\frac{(1-2k)\cdot\frac{b}{2}+k\cdot\frac{cb}{a+c}}{(1-2k)\cdot\frac{b}{2}+k\cdot\frac{ab}{a+c}}=\frac{a+c-2ka}{a+c-2kc}.$$

Let us find k. By S denote the foot of bisector BI in triangle ABC. We get

$$k=\frac{BL_2}{BS}=\frac{BL_2/BI}{BS/BI}=\frac{\cos\beta}{1+IS/BI}=\frac{\cos\beta}{1+AS/AB}=\frac{\cos\beta}{1+\frac{cb}{c(a+c)}}=\frac{(a+c)\cos\beta}{a+b+c}.$$

Taking into account that $\cos\beta=\frac{a^2+c^2-b^2}{2ac}$, we obtain

$$\frac{C_0N}{A_0N}=\frac{a+c-2\frac{(a+c)(a^2+c^2-b^2)}{2ac(a+b+c)}a}{a+c-2\frac{(a+c)(a^2+c^2-b^2)}{2ac(a+b+c)}c}=\frac{a(a+b)(c+b-a)}{c(b+c)(a+b-c)}.$$

The ratio for the other sides of the medial triangle equals

$$\frac{b(b+c)(a+c-b)}{a(c+a)(b+c-a)} \text{ and } \frac{c(c+a)(b+a-c)}{b(a+b)(c+a-b)}, \text{ respectively.}$$

It can easily be checked that the product of these three fractions equals 1. This completes the proof.