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Three mysterious points of triangle

Solution of the Main Part

11. Let $H_1'H_2'H_3'$ be the orthic triangle of the intouch triangle (fig. 15).

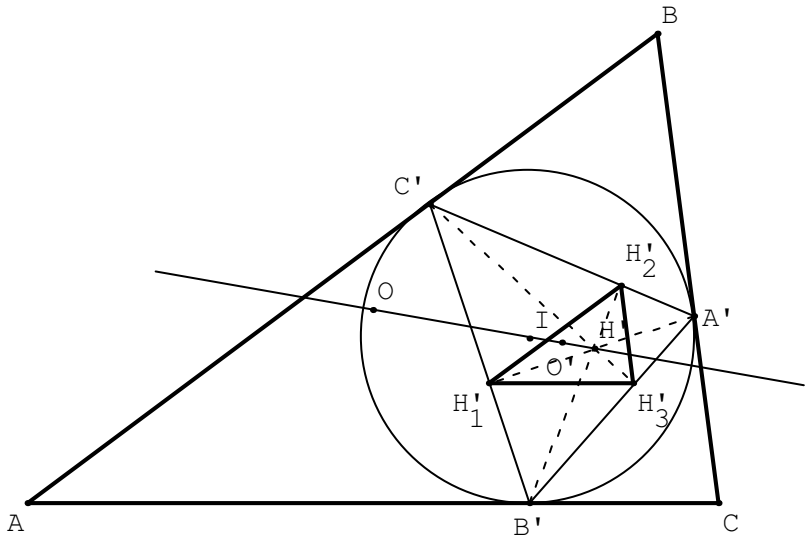

Fig. 15

It is easy shown that $H_1'A', H_2'B', H_3'C'$ are bisectors of $\Delta H_1'H_2'H_3'$ and H' is the incenter of $\Delta H_1'H_2'H_3'$. Further, $\Delta H_1'H_2'H_3'$ is homothetic to ΔABC. This homothety takes H' to I, O' to O, here O' is the circumcenter of $\Delta H_1'H_2'H_3'$ (note that the circumcircle of $\Delta H_1'H_2'H_3'$ is the nine-point circle of the intouch triangle). Thus $H'I$ and $O'O$ pass through the center of homothety. O', H', I are collinear (O', H', I lie on the Euler line of $\Delta A'B'C'$), therefore O lies on $H'I$.

Another solution of Problem 11 (A. Ganbat)

The invertion with respect to the incircle of ΔABC takes A, B, C to the midpoints of the segments $B'C'$, $C'A'$, $A'B'$, respectively. Hence this invertion takes the circumcircle of ΔABC to the nine-point circle of the intouch triangle. This implies that the points O, O' (the center of the nine-point circle of the intouch triangle), and I are collinear. To conclude the proof, it remains to note that $O'I$ is the Euler line of $\Delta A'B'C'$.

12 a) By A'', B'', C'' denote the midpoints of $\Delta A'B'C'$ (fig. 16).

$\Delta A''B''C''$ is homotetic to $\Delta A'B'C'$ with respect to its centoid M' with ratio $-\frac{1}{2}$. From the other hand, $\Delta A''B''C''$ is homothetic to $\Delta L_1L_2L_3$ with respect to I with ratio $\frac{1}{2}$. Further, $\Delta A'B'C'$ and $\Delta L_1L_2L_3$ are homothetic with ratio -1. The product of three pairwise ratio of homothety is positive, therefore three centers of homothety lie on a line. This implies that the homothety center of $\Delta L_1L_2L_3$ and $\Delta A'B'C'$ lies on IM'. Note that IM' is the Euler line of $\Delta A'B'C'$, thus IM' coincides to IO.

b) The line passing through the circumcenters of two homothetic triangles passes through the center of homothety. I is the circumcenter of $\Delta A'B'C'$. From a) it follows that the circumcenter of $\Delta L_1L_2L_3$ lies on IO.

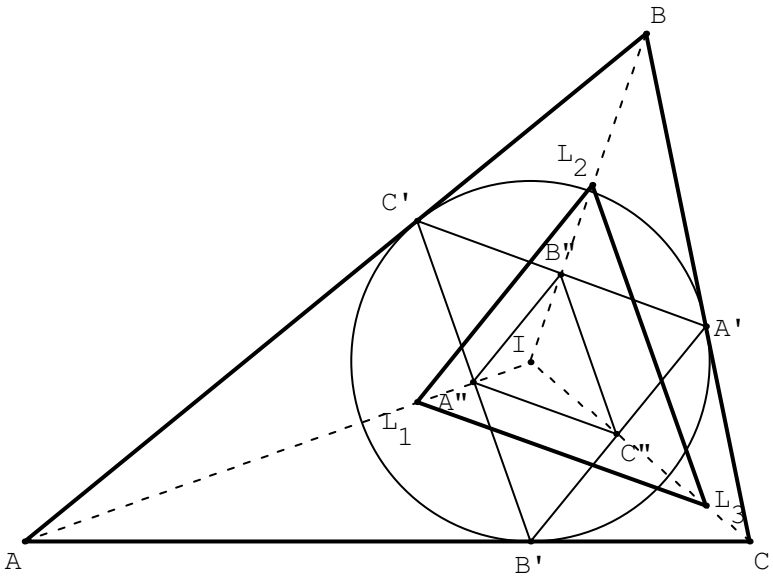

Fig. 16

Another solution of Problem 12. $\Delta L_1L_2L_3$ and $\Delta A'B'C'$ are symmetric with respect to a point. The orthocenter of one of them is the circumcenter of the other. Hence these triangles have the common Euler line; moreover, the symmetry center of these triangles is the midpoint of the segment H'I, thus the center of symmetry coincides to the center of the nine-point circle of $\Delta A'B'C'$ and $\Delta L_1L_2L_3$.

13. The perpendicular bisector of AL_1 passes through O_2 and O_3 (fig. 17), hence $O_2O_3 \parallel B'C' \parallel L_2L_3$. Similarly, $O_1O_3 \parallel L_1L_3$, $O_1O_2 \parallel L_1L_2$. Therefore $\Delta O_1O_2O_3$ and $\Delta L_1L_2L_3$ are homothetic.

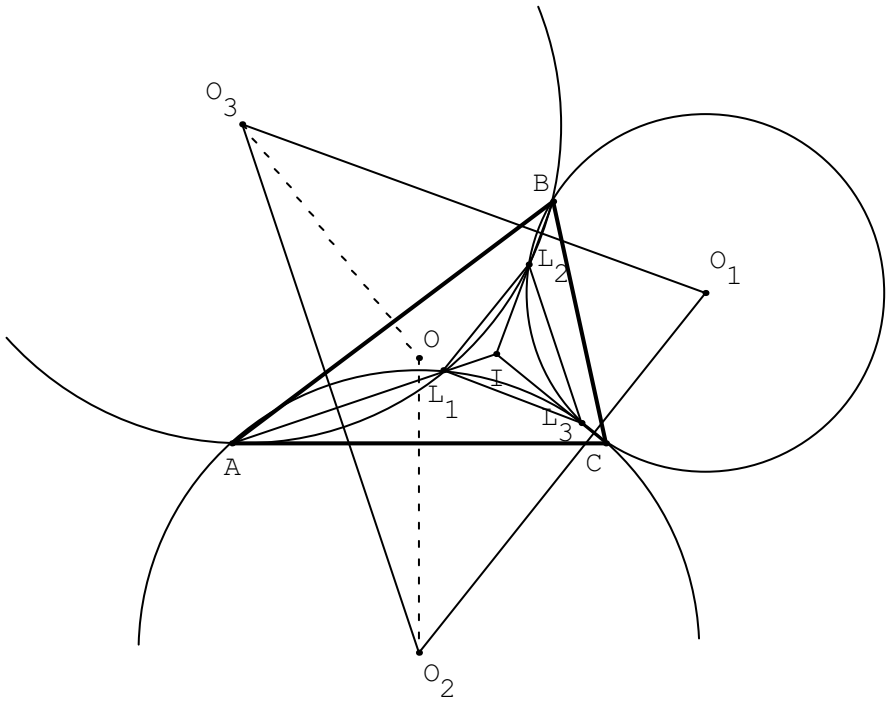

Fig. 17

14. The perpendicular bisector of AB passes through O_3 and O, the perpendicular bisector of AC passes through O_2 and O (fig. 17).

$\angle O_2OO_3 = 180° - \alpha = \beta + \gamma$, this implies that $\angle O_2O_1O_3 = \angle B'A'C' = \frac{\beta+\gamma}{2} = \frac{1}{2}\angle O_2OO_3$. Similarly, $\angle O_1O_2O_3 = \frac{1}{2}\angle O_1OO_3$ и $\angle O_2O_3O_1 = \frac{1}{2}\angle O_2OO_1$. It follows easily that O_1, O_2, and O_3 lie on a circle with center O.

15. $\Delta A'B'C'$ and $\Delta O_1O_2O_3$ are homothetic with positive ratio; the center of this homothety lies on line IO connecting the circumcenters of $\Delta A'B'C'$ and $\Delta O_1O_2O_3$.

$\Delta A'B'C'$ and $\Delta L_1L_2L_3$ are homothetic with ratio –1; the center of this homothety lies on IO (Problem 12).

$\Delta O_1O_2O_3$ and $\Delta L_1L_2L_3$ are homothetic with negative ratio. From the Theorem about three centers of homothety it follows that the homothetic center of $\Delta O_1O_2O_3$ and $\Delta L_1L_2L_3$ lies on IO.

16. Since the perpendicular bisector of L_2L_3 passes through A' and O_1, $\Delta A'B'C'$ and $\Delta O_1O_2O_3$ have common altitudes. Hence the common orthocenter H' of $\Delta A'B'C'$ and $\Delta O_1O_2O_3$ is the center of homothety that takes $\Delta A'B'C'$ to $\Delta O_1O_2O_3$. The ratio of this homothety equals $k = \frac{H'O}{H'I} = 1 + \frac{IO}{H'I}$.

The center O' of the nine-point circle of $\Delta A'B'C'$ is the midpoint of $H'I$. From solution of problem 11 it follows that the homothetic ratio of ΔABC and $\Delta H_1'H_2'H_3'$ equals $\frac{IO}{H'O'}$. From the other hand, this ratio is equal to the ratio $\frac{R}{r/2}$ of radii of the circumcircles of ΔABC and $\Delta H_1'H_2'H_3'$.

Finally, $k = 1 + \dfrac{IO}{H'I} = 1 + \dfrac{IO}{2H'O'} = 1 + \dfrac{R}{r} = \dfrac{r+R}{r}$. This implies that the circumcenter of $\Delta O_1O_2O_3$ is of radius $R + r$.

17. (O. Golberg) For example, let us prove that $A_1B_1 \parallel AB$. AB is the radical axis of (ABC) and ω_3. Note that (ABC) and $(O_1O_2O_3)$ are concentric, hence $A_1B_1 \parallel AB$.

Then, let us prove that A_1 lies on AI. A_1 is the radical center of ω_2, ω_3, $(O_1O_2O_3)$, since A_1 lies on the radical axis of the pairs of circles $\omega_2, (O_1O_2O_3)$ and $\omega_3, (O_1O_2O_3)$. Therefore A_1 lies on the radical axis AI of the circles ω_2 and ω_3.

Find the ratio of homothety as the ratio of the distances from the homothetic center I to B_1C_1 and BC (note that the second distance equals r). By W denote the intersection point of B_1C_1 and OO_1, by T denote the projection of I on OO_1 (fig. 18).

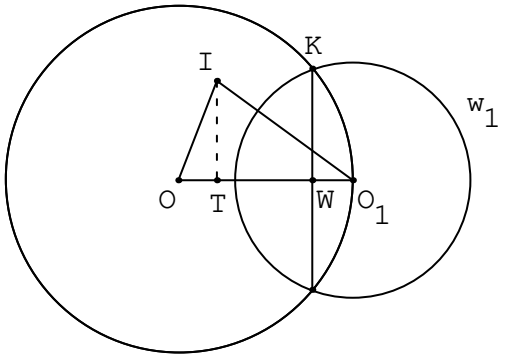

Fig. 18

We put $O_1W = m$, $WO = n$, $\pm WT = t$, here we take «+» if $T \in [WO)$ and we take «–» if $T \in [WO_1)$. The distance from I to B_1C_1 equals $|t|$. We have

$$IO_1^2 - IO^2 = O_1T^2 - OT^2 = (m+t)^2 - (n-t)^2 = (m^2 - n^2) + 2(m+n)t,$$

$$m^2 - n^2 = KO_1^2 - KO^2 = R_1^2 - (R+r)^2, \text{ here } R_1 \text{ is radius of } \omega_1,$$

$$IO_1^2 - IO^2 = R_1^2 - (R+r)^2 + 2(m+n)t,$$

$$IO_1^2 - R_1^2 - IO^2 + (R+r)^2 = 2(m+n)t.$$

$IO_1^2 - R_1^2$ is the power of I with respect to ω_1. I is the radical center of ω_1, ω_2, ω_3, hence I has equal powers with respect to ω_1, ω_2, ω_3. One can easily calculate that this power equals $IL_1 \cdot IA = 2r^2$. Therefore $2(m+n)t = 2r^2 - IO^2 + (R+r)^2$. Taking into account that $IO^2 = R^2 - 2Rr$ (*Euler*), we get

$$t = \frac{2r^2 + (R+r)^2 - (R^2 - 2Rr)}{2(m+n)} = \frac{3r^2 + 4Rr}{2(m+n)}.$$

The ratio of homothety is equal to

$$\frac{t}{r} = \frac{3r + 4R}{2R + 2r} = \frac{3 + 4\,R/r}{2 + 2\,R/r} = 2 - \frac{1}{2 + R/r}.$$

Since $\dfrac{R}{r}$ could take any value from $[2, \infty)$ (this follows from the Euler formula), we obtain that the ratio of homothety varies in $\left[\dfrac{11}{6};\ 2\right)$.

18. (Fig. 19)

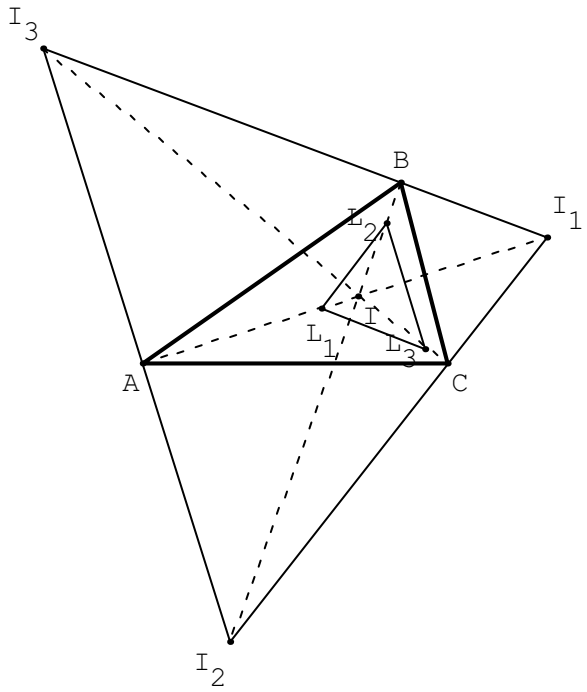

Fig. 19

$I_2I_3 \perp AI$ as interior and exterior bisectors ΔABC. Hence $I_2I_3 \parallel L_2L_3$. This implies that triangles $I_1I_2I_3$ and $L_1L_2L_3$ are homothetic.

L_1 and I_1 lie on AI, therefore I is the center of homothety.

19. The triangles $I_1I_2I_3$ and $O_1O_2O_3$ are homothetic since each of them is homothetic to $\Delta L_1L_2L_3$. The triangles $I_1I_2I_3$ and $L_1L_2L_3$ are homothetic with center I and negative ratio. $\Delta O_1O_2O_3$ and $\Delta L_1L_2L_3$ are homothetic with negative ratio and center lying on IO (Problem 15). Therefore the triangles $I_1I_2I_3$ and $O_1O_2O_3$ are homothetic with positive ratio and center lying on IO (by Theorem about three centers of homothety).

20. By M_1, M_2, M_3 denote the pairwise homothetic centers of three circles (fig. 20). L_2L_3 is the radical axis of the circles $(L_1L_2L_3)$ and ω_1. BC is the radical axis of (ABC) and ω_1. Hence the point M_1 of intersection of these radical axes is the radical center of $(L_1L_2L_3)$, (ABC) и ω_1, hence M_1 lies on the radical axis of $(L_1L_2L_3)$ and (ABC). Similarly, M_2 and M_3 lie on this radical axis. This is wee known that the radical axis is perpendicular to the center line of two circles. Therefore t is perpendicular to $H'O$ (that is IO).

<u>*Another solution of Problem 20*</u> (Kulikov A.) By Theorem about three centers of homothety M_1, M_2, M_3 lie on a line (fig. 20).

The pairwise extangents intersect at H' (Problem 5) that lies on IO. Let us prove that $M_1M_2 \perp IH'$. By inverse Pythagorean proposition $M_1M_2 \perp IH'$ is equivalent to equality $H'M_2^2 - H'M_1^2 = IM_2^2 - IM_1^2$. We have

$$IM_2^2 - IM_1^2 = \left(B'I^2 + B'M_2^2\right) - \left(A'I^2 + A'M_1^2\right) = B'M_2^2 - A'M_1^2.$$

B' is symmetrical to H' with respect to L_1L_3 (Problem 5), hence $B'M_2 = H'M_2$. Similarly, we get $A'M_1 = H'M_1$. Thus $B'M_2^2 - A'M_1^2 = H'M_2^2 - H'M_1^2$.

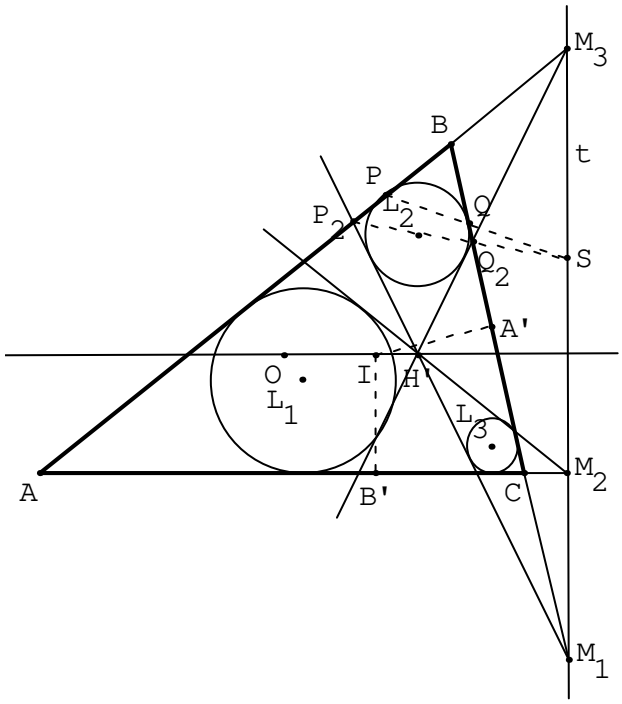

Fig. 20

21. Three lines passing through the opposite vertices of an exscribed hexagon have a common point (Bryanshon). Therefore lines M_1M_3, PQ, P_2Q_2 pass through a common point (point *S* on fig. 20) being the diagonals of the exscribed hexagon $M_1P_2PM_3Q_2Q$.

22. *Lemma.* Suppose *X, Y, Z, X′, Y′, Z′* are points such that the sum $(XZ',YZ')+(YX',ZX')+(ZY',XY')$ is divisible by π; then the circles *(XYZ′), (YZX′), (ZXY′)* have a common point. (Here by (a,b) we denote *an oriented angle* between lines *a, b.* Let us recall the definition of an oriented angle. Let us rotate *a* counter clockwise until *a* coincides to *b.* The angle of rotation is called the oriented angle between *a* and *b;* it is defined up to adding of πk. The basic properties: $(a,b)=-(b,a)$, $(a,b)+(b,c)=(a,c)$; suppose *A* and *B* are fixed points , then a circle passing through *A* and *B* is a locus of *X* such that (AX,BX) is equal to a constant.)

Proof of Lemma.

If any two of the circles coincide, then the proof is obvious.

By *P* denote the second intersection (not *Y*) point of the circles *(XYZ′)* and *(YZX′).* We have $(XP,YP)=(XZ',YZ')$ и. $(YP,ZP)=(YX',ZX')$.

(If *P* coincides to one of the points *X*, *Z*, then the proof is clear; if *(XYZ′)*, and *(YZX′)* have a touch point *Y*, then we put *P=Y*, in this case by *YP* denote the common tangent to the circles *(XYZ′)*, *(YZX′)* passing through *Y*.)

This implies that $(XP,ZP)=(XP,YP)+(YP,ZP)=(XZ',YZ')+(YX',ZX')=-(ZY',XY')=(XY',ZY')$. From that it follows that *X, Y′, Z, P* are concyclic.

Solution of Problem 22.

We have

$(A'L_3,L_3B')+(B'L_1,L_1C')+(C'L_2,L_2A')=(A'I,IB')+(B'I,IC')+(C'I,IA')=2\pi$, therefore from Lemma it follows that $(L_1B'C')$, $(L_2C'A')$, $(L_3A'B')$ have a common point. Now let us prove that $(L_1L_2L_3)$, $(L_2C'A')$, $(L_3A'B')$ have a common point:

$$(L_3L_1,L_1L_2)+(L_2C',C'A')+(A'B',B'L_3)=\left(\frac{\beta+\gamma}{2}\right)+\left(\pi-\frac{\beta}{2}\right)+\left(\pi-\frac{\gamma}{2}\right)=2\pi.$$

In the same way we get that any of four triples of the circles have a common point.

If one of four circles coincides to another or touches another, then it easily follows that all four circles have a common point. In the other case from the proof of Lemma it follows that $(L_1B'C')$, $(L_2C'A')$, and

$(L_3A'B')$, have a common point P that does not coincide to A', B', C'. Suppose $(L_1L_2L_3)$ does not pass through P, then it must pass through A', B', C', thus all four circles coincide to each other.

We see that in all cases four circles have a common point.

From the symmetry of triangles $L_1L_2L_3$ and $A'B'C'$ the second statement of Problem follows.

23. We have

$(H_1L_3, L_3H_2)+(H_2L_1, L_1H_3)+(H_3L_2, L_2H_1)=\left(\pi-\frac{\alpha+\beta}{2}\right)+\left(\pi-\frac{\beta+\gamma}{2}\right)+\left(\pi-\frac{\gamma+\alpha}{2}\right)=2\pi$, therefore from Lemma it follows that $(L_1H_2H_3)$, $(L_2H_3H_1)$, $(L_3H_1H_2)$ have a common point. Now let us prove that $(L_1L_2L_3)$, $(L_2H_3H_1)$, $(L_3H_1H_2)$ have a common point.

$$(L_3L_1, L_1L_2)+(L_2H_3, H_3H_1)+(H_1H_2, H_2L_3)=\left(\frac{\beta+\gamma}{2}\right)+\left(\pi-\frac{\gamma}{2}\right)+\left(\pi-\frac{\beta}{2}\right)=2\pi.$$

In the same way we get that any of four triples of the circles have a common point.

Further by the reasoning similar to that in the proof of Problem 22 it follows that all four circles have a common point.

24. We know that A' is symmetrical to H' with respect to L_2L_3. We get

$$(A'L_2, A'L_3)=(H'L_3, H'L_2)=2\cdot(L_1L_3, L_1L_2)=\beta+\gamma.$$

$$(H_1L_2, H_1L_3)=\pi-\frac{\alpha}{2}-\frac{\alpha}{2}=\beta+\gamma.$$

Hence H_1 lies on the circle ($A'L_2L_3$). Аналогично $H_2\in(B'L_1L_3)$ и $H_3\in(C'L_1L_2)$.

These three circles have a common point lying on the incircle (Problem 22). Let us prove that also this point lies on the nine-point circle. We have

$$(L_1L_3, L_3H_2)+(H_2H_1, H_1H_3)+(H_3L_2, L_2L_1)=\frac{\pi}{2}-\alpha+2\alpha+\frac{\pi}{2}-\alpha=\pi$$

From Lemma it follows that $(H_2L_1L_3)$, $(H_3L_1L_2)$, and $(H_1H_2H_3)$, have a common point. This point is the Feuerbach point (that is a unique coomon point of the incircle and the nine-point circle).

25. For example, let us prove that $(BA, BC)=(S_2S_3, S_2S_1)$ (fig. 21).

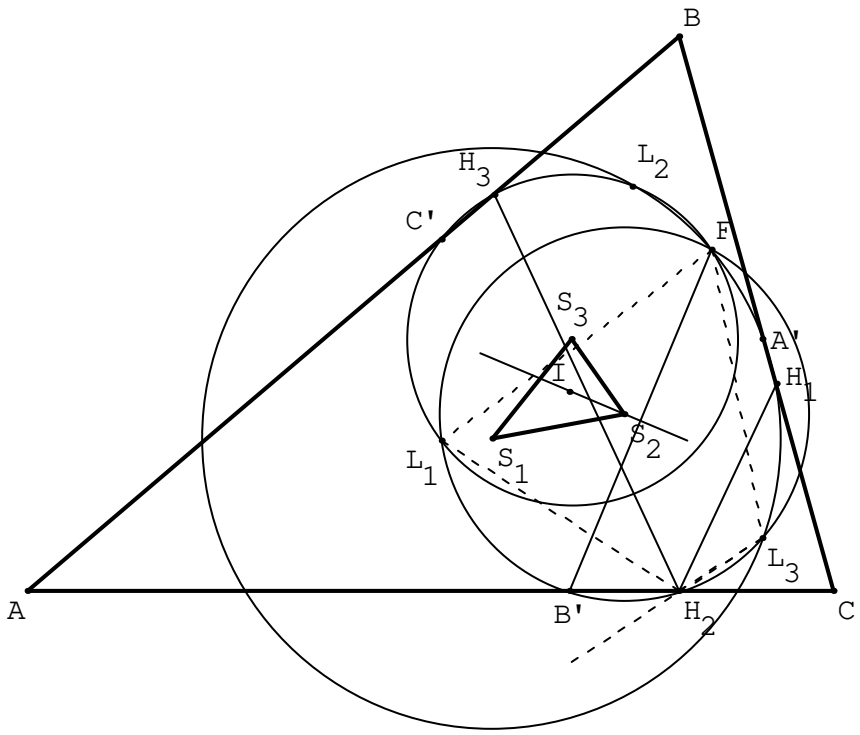

Fig. 21

FL_1 is the radical axis of $(L_1H_3L_2)$ and $(L_1H_2L_3)$, hence $FL_1 \perp S_2S_3$. Similarly, $FL_3 \perp S_1S_2$. Therefore

$$(S_2S_3, S_2S_1) = (FL_1, FL_3) = (H_2L_1, H_2L_3) = (H_2C, H_2L_3) + (H_2L_1, H_2A).$$

L_1 and L_3 are the incenters of the triangles AH_3H_3 and CH_2H_1, hence

$$(H_2C, H_2L_3) + (H_2L_1, H_2A) = \frac{1}{2}(H_2C, H_2H_1) + \frac{1}{2}(H_2H_3, H_2A) = (BA, BC).$$

26. (Fig. 21) $B'L_1 = B'L_3$ (each of these segments equals r), therefore FB' is the bisector of angle (FL_1, FL_3). FB' is a common chord of $(L_1H_2L_3)$ and the incircle. The perpendicular bisector of FB' passes through the centers S_2 and I of these circles. From the solution of Problem 25 it follows that the bisectors of the angles (S_2S_3, S_2S_1) and (FL_1, FL_3) are perpendicular. Therefore the perpendicular bisector m of FB' is the bisector of (S_2S_3, S_2S_1). Since m passes through I, it follows that I is the incenter of $\Delta S_1S_2S_3$.

27. <u>Lemma</u> For triangle $X_1X_2X_3$, there exists at least one point P (distict from X_1, X_2, X_3) such that (PX_1, PX_2) and (PX_2, PX_3) are equal to fixed constants.

<u>*Proof of Lemma*</u>

The locus of P such that (PX_1, PX_2) equals a fixed constant, is a circle. Similarly, the locus of P such that (PX_2, PX_3) equals a fixed constant, is a circle. These two circles have at least two common points one of which coincides to X_2.

<u>*Solution of Problem 27.*</u>

If E coincides to one of the points S_1, S_2, S_3, it follows easily that S_1, S_2, S_3 coincide.

Let E be the circumcenter of $\Delta A_0B_0C_0$ (recall that $(A_0B_0C_0)$ coincides to $(H_1H_2H_3)$). H_1F, H_2F, and L_3F, are the pairwise radical axes of the circles $(H_1H_2H_3)$, $(H_1L_2L_3)$, $(H_2L_1L_3)$. Therefore we have

$$\left(ES_1, ES_2\right) = \left(FH_1, FH_2\right) = \left(H_3H_1, H_3H_2\right) = \left(H_3H_1, H_3B\right) + \left(H_3A, H_3H_2\right) =$$
$$= 2\left(CB, CA\right) = 2\left(C_0B_0, C_0A_0\right) = \left(EB_0, EA_0\right).$$

Similarly, $\left(EC_0, EA_0\right) = \left(ES_1, ES_3\right)$. Note that triangles $A_0B_0C_0$ and $S_1S_2S_3$ are similar. Finally, from Lemma it follows that E is the circumcenter of the triangle $S_1S_2S_3$.

Addition to Problems 25, 26, 27

It could be found a transformation that takes ΔABC to $\Delta S_1S_2S_3$. This transformation is a composition of homothety with center I and ratio $-\frac{1}{2}\cdot\sqrt{1-2\frac{r}{R}}$, and symmetry with respect to the bisector of the angle OIF.

Appendix. List of Results.

28. (A. Kulikov) $\overline{IX} = -\frac{2r}{R-r}\cdot\overline{IO}$, here X is the homothetic center from Problem 19.

29. (D. Shiryaev) Let A^*, B^*, C^* be the intersection points of the bisectors of ΔABC and the circumcircle. By Q denote the center of the nine-point circle of $\Delta A^*B^*C^*$. The Schiffler point of ΔABC is isogonal conjugate of Q with respect to $\Delta A^*B^*C^*$.

32. The lines passing through the vertices of ΔABC that are perpendicular to the corresponding sidelines of $\Delta L_1L_2L_3$ intersect at I.

33. The lines passing through the vertices of $\Delta L_1L_2L_3$ that are perpendicular to the corresponding sidelines of ΔABC intersect at the center of the nine-point circle of the intouch triangle.

34. The lines passing through the vertices of $\Delta L_1L_2L_3$ that are perpendicular to the corresponding sidelines of the Nagel triangle intersect at H.

35. Suppose the lines passing through the vertices of the Nagel triangle that are perpendicular to the corresponding sidelines of $\Delta L_1L_2L_3$ intersect at *P*. *P* is symmetric to the orthocenter of the intouch triangle with respect to the Spieker point. (*The Spieker point* is the incenter of the medial triangle.)