INTERLOCKING STRUCTURES

Here we present the set of problems concerning flat and spatial configurations with interesting properties.

A. Preparatory Problems

A1. There is a finite number of convex figures on the table. Prove that one of them may be shifted. We say that figure/solid can be *shifted* if one can make a translation in some direction to any given distance. (Here we consider only translations, not rotations.)

A2. There is a finite number of spheres in the space. Prove that one of them can be shifted (in the way discussed above).

A3. Prove the same for n-dimensional space. (If there are several spheres in n-dimensional space then there exists a sphere that can be shifted)

A4. There are several nonintersecting round chocolates lay on a round cake. Prove that one can divide the cake into convex parts, each part containing exactly one chocolate.

A5. Is the previuos statement true for triangular chocolates?

A6. There is a set of nonintersecting small balls inside a big one B. Is it always true that B can be split into convex parts each one containing exactly one small ball?

A7. Is it possible to place congruent convex solids in a layer between two parallel planes so that none of them could be shifted?

A8. Is it possible to place congruent convex solids in a layer between two planes so that no finite number of them could be shifted?

A9. Prove that any system of convex solids can be moved apart if you move all solids simultaneously.

A10. The same for star-like solids. A solid is called *star-like* if all its surface is visible from some internal point.

B. Translations and Rotations

Let us say that the system of solids is *translation locked* if no solid can be shifted.

Let us say that the system of solids is *rotation locked* if no solid can be shifted even if you are allowed to use both rotations and translations.

B1. Is it possible to costruct a translation locked system of a solids which is not rotation locked?

B2. The same question for a big convex solid treated like a system of convex solids.

C. When a Block is Locked? To formulate the criterium we need some elementary geometric facts.

C1. Prove that any line on the plane Oxy consists of points (x, y) such that $ax + by = c$ for some a, b, c.

C2. Prove that any plane in the space $Oxyz$ consists of points x, y, z, such that $ax + by + cz = d$ for some a, b, c, d.

C3. Prove that the vector (a, b) is orthogonal to the line $ax + by = c$ and the vector (a, b, c) — to the plane $ax + by + cz = d$.

C4. Prove that any convex polygon/polyhedron is the intersection of the semispaces with boundaries containing its edges/faces.

Let each of lines containing an edge of a polygon on the plane move in the orthogonal direction. This can be regarded as an evolution of the polygon. (The number of edges can change and the polygon can degenerate into a segment or a point. By definition, it forces the evolution to stop.) Let v_i be the velocity of i-th line.

C5. Find an example when both the evolution with velocities v_i and the one with velocities $-v_i$ lead to the degeneration.

C6. Prove that polygon M degenerates if and only if any of the following situations occurs:

1) Any triangle whose edges contain three M's edges, dehenerates.

2) During the evolution, the distance between two parallel lines that contain M's edges decreases.

C7. Generalize these statements to the dimensions > 3.

Consider a polygon or a polyhedron. The set of velocities v_i is called *degenerative* if this polygon (or polyhedron) degenerates with this set of velocities. The set $\{v_i\}$ is called *strictly degenerative* if it is degenerative and $\{-v_i\}$ is also degenerative.

C8. Find strictly degenerative sets for a) square; b) pentagon; c) regular n-gon; d) cube; e) octahedron.

Let $\mathit{\Pi}$ be a semispace of points (x, y, z) such that $z \geq 0$. Suppose that $\mathit{\Pi}$ is filled with convex polyhedrons. Let M be one of the polyhedrons and $\mathit{\Gamma}$ be one of its faces. We want to describe all possible directions l such that M can be shifted along l.

Let $a_i x + b_i y + c_i z = d_i$ be an equation of plane α_i contains the i-th face of M which has common edge e_i whith face $\mathit{\Gamma}$. Let φ_i be an angle between Γ and α_i. Let all points of M satisfy condition $a_i x + b_i y + c_i z \leq d_i$.

C9. Prove that the vectors (V_1, V_2, V_3) of possible shifts of M form a polyhedral cone determined by a system of inequalities $V_3 \leq 0$, $a_i V_1 + b_i V_2 + c_i V_3 \leq 0$.

In particular, if such system has no solutions then M is locked.

Express $\cos\varphi_i, \sin\varphi_i$ in terms of a_i, b_i, c_i.

C10. Let us consider a polyhedron M' such that all planes containing faces of M' coincide with those of M except $\{z = 0\}$. The plane containing a new face Γ' of M' is a result of a shift of the plane containing Γ along Oz on a

distance Δt. Prove that Γ' is congruent to the polygon that is obtained from Γ by shifting i-th edge on $\Delta t \operatorname{ctg} \varphi_i$ in perpendicular direction (for all i).
C11. Let ith side of Γ move with velocity $V_i = \operatorname{ctg} \varphi_i$. Prove that M is locked if and only if Γ degenerates during the evolution.

D. Regular Polygons

Does there exist an arrangement of pairwise congruent polyhedrons in a layer between two parallel planes such that:

Problem A. It is impossible to shift any of polyhedrons.

Problem B. It is impossible to shift any finite number of polyhedrons.

Solve these problems for following polyhedrons:
D1. Tetrahedrons.
D2. Octahedrons.
D3. Cubes.
D4. Icosahedrons.
D5. Truncated icosahedrons (football balls).
D6. Dodecahedrons.
D7. Solve these problems for 4-dimensional cubes in a layer between two parallel 3-dimensional hyperplanes.

E. Lattices.

E1. Consider a standard 3-dimensional rectangular lattice. We choose a cube K and draw a plane of type $x + y + z = c$ that contains the center of K. Describe the intersection of this plane with the lattice.
E2. The same question for 4-dimesional rectangular lattice.

F. Basic Constructions.

F1. Does there exist a partition of a layer between two parallel lines into convex figures F_i such that:

1) No F_i can be shifted;

2) The number of possible shapes is finite. (It means that there exist figures $W_1, \dots, W_n$ such that each F_i is congruent to one of W_j's.)
F2. The same question for a partition of a bounded convex solid into several convex solids.
F3. Let B be a segment of a ball[1] which is strictly less than a semiball. Does there exist a partition of B into convex solids such that :

1) Diameter of each B_i is less than $1/1000$ height of B.

Let Γ be a B's plane face. The part B_i is *interesting* if it has a common point with Γ.

2) Whenever all interesting parts are fixed, all other parts can not be shifted (translations may depend on B_i).
F4. The same question for a segment which contains a semiball.
F5. Formulate and solve problems similar to **F1** and **F2** for star-like plane figures.

Here we begin to discuss stably locked arrangments.

The arrangement $I = \{F_i\}$ of convex solids is *locked* if each F_i cannot be shifted whenever all others are fixed.

The arrangement $I = \{F_i\}$ of convex solids is *stably locked* if for each finite set H of F_i's there exists a finite set $H' \supseteq H$ such that the arrangement $I \setminus H'$ is locked.

The arrangement $I = \{F_i\}$ *contains finite numbers of shapes* if there exists a finite subset $\{W_j\}$ such that each F_i is congruent to one of W_j's.
F6. Does there exist a stable partition of a semispace into convex solids of finite number of shapes?
F7. The same questions for a layer between two parallel planes.

[1]By a segment of a ball we mean the intersection of a ball with a semispace.