How to derive a number?

Solution of problems 1-6, 11.

Problem 1. There we have an explicite formula for the derivative:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k} \Rightarrow n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$$

Let us show that Leibniz rule holds. If $n = n_1 n_2$, $n_1 = p_1^{\beta_1} \cdot \ldots \cdot p_k^{\beta_k}$, $n_2 = p_1^{\gamma_1} \cdot \ldots \cdot p_k^{\gamma_k}$, $\alpha_i = \beta_i + \gamma_i$, then

$$n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i} = n_1 n_2 \sum_{i=1}^{k} \frac{\beta_i + \gamma_i}{p_i} = \left(n_1 \sum_{i=1}^{k} \frac{\beta_i}{p_i} \right) n_2 + n_1 \left(n_2 \sum_{i=1}^{k} \frac{\gamma_i}{p_i} \right) = (n_1 n_2)'.$$

The formula defines a function. To prove that the derivative is unique let us show with induction by n that any derivative coincide with the function. That is true for $n = 1$. Assume that we have proved the coincidence for integers less then n. If n is a prime, the coincidence is also evident. Otherwise we can factorize n. As both factors are less then n, a derivative of them coincide with the function. Using the Leibniz rule to derive n, we can se as before that the derivative be the same as the formula result, i.e. a derivative and the function coincide on n.

Problem 2. Let us rewrite both the right and the left part:

$$(ka + kb)' = (k(a + b))' = k'(a + b) + k(a + b)'$$

$$(ka)' + (kb)' = k'a + ka' + k'b + kb' = k'(a + b) + k(a' + b')$$

Now it is clear that we can compare only the terms $k(a + b)'$ and $k(a' + b')$. As $k > 0$, we get the same inequality as it have been for terms $(a + b)'$ and $a' + b'$.
By the way, we have proved also the third property:

$$(a + b)' = a' + b' \Rightarrow (ka + kb)' = (ka)' + (kb)'.$$

Problem 3. These equalities and inequalities can be easily derived from the assertion in Problem 2 and from the identities a) $3' = 1' + 2'$, b) $2' \geqslant 1' + 1'$, c) $5' \leqslant 3' + 2'$;

The other way is to derivate directly: $(3k)' = 3k' + 3'k = 3k' + k = k' + (2k' + 2'k) = k' + (2k)'$.

Problem 4. If n is a prime, then $n' = 1$; if it can be represented as a product of 3 or more factors, then $n' \geqslant (p_1p_2 + p_1p_3 + p_2p_3) \geqslant 12$. Thus, if $2 \leqslant a \leqslant 10$, $n = p_1p_2$ and $n' = p_1 + p_2$; to solve the equation for such a, we must try to represent them as a sum of two primes: $2 \neq p_1 + p_2$; $3 \neq p_1 + p_2$; $4 = 2 + 2,\ 4' = 4$; $5 = 2 + 3,\ 6' = 5$; $6 = 3 + 3,\ 9' = 6$; $7 = 2 + 5,\ 10' = 7$; $8 = 3 + 5,\ 15' = 8$; $9 = 2 + 7,\ 14' = 9$; $10 = 3 + 7 = 5 + 5,\ 21' = 25' = 10$.

Problem 5. Let's represent a number as a product of primes, some of them may be same $n = p_1 p_2 \cdot \ldots \cdot p_k$.
Upper estimation. As $p_i \geqslant 2$, $n \geqslant 2^k$, so $k \leqslant \log_2 n$. Now

$$n' = n \sum_{i=1}^{k} \frac{1}{p_i} \leqslant n \sum_{i=1}^{k} \frac{1}{2} \leqslant nk\frac{1}{2} \leqslant n\frac{\log_2 n}{2}$$

Lower estimation.

$$n' = n\sum_{i=1}^{k}\frac{1}{p_i} = nk\frac{\sum\limits_{i=1}^{k}\frac{1}{p_i}}{k} \geqslant nk\sqrt[k]{\prod_{i=1}^{k}\frac{1}{p_i}} = nk\sqrt[k]{\frac{1}{n}} = kn^{\frac{k-1}{k}}$$

If n is a composite number, then $k \geqslant 2$ and $n' > 2\sqrt{n}$. Note that both estimations are exact for infinitly many numbers $n = 2^k$.

Problem 6. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k}$, then

$$n' = \alpha_1 p_1^{\alpha_1-1}p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k} + \alpha_2 p_1^{\alpha_1}p_2^{\alpha_2-1}\cdot\ldots\cdot p_k^{\alpha_k} + \ldots + \alpha_k p_1^{\alpha_1}p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k-1}$$

All of terms except first one are divisible by $p_1^{\alpha_1}$, hence α_1 is divisible by p_1. If $\alpha_1 > p_1$, then the first term would be greater then n, and other terms are nonnegative. Thus, $\alpha_1 = p_1$. So, the first term equals n, and others must equal zero. That's why $\alpha_2 = \alpha_3 = \ldots = \alpha_k = 0$ and $n = p_1^{p_1}$.

Problem 11. Let as assume that we have managed to extend the derivative to rational numbers. Then by derivation of equalities $(x^2)' = ((-x)^2)'$ and $m' = \left(\frac{m}{n}\cdot n\right)'$ using Leibniz rule one can easyly derive that the derivative commutate with minus and the usual rule to derivate a fraction holds. So the extension must be unique.

Let us build such en extension. We define for an irreducible fraction $\frac{m}{n}$ (where $n > 1$) that

$$\left(\frac{m}{n}\right)' = \frac{m'n - mn'}{n^2}.$$

Now observe that the result will be the same if we represent a positive integer as a fraction with the denomintor 1. Let us check that the formula for fractions give the correct result even when it for a reducible fraction:

$$\left(\frac{km}{kn}\right)' = \frac{(km)'kn - km(kn)'}{k^2n^2} = \frac{k'mkn + km'kn - kmk'n - kmkn'}{k^2n^2} = \frac{m'n - mn'}{n^2}.$$

Now we can check that Leibniz rule holds for a product of two positive rational numbers:

$$\left(\frac{m}{n}\cdot\frac{p}{q}\right)' = \frac{(mp)'nq - mp(nq)'}{(nq)^2} = \frac{(m'p + mp')nq - mp(n'q + nq')}{(nq)^2} =$$

$$= \frac{m'n - mn'}{n^2}\cdot\frac{p}{q} + \frac{p'q - pq'}{q^2}\cdot\frac{m}{n} = \left(\frac{m}{n}\right)'\cdot\frac{p}{q} + \frac{m}{n}\cdot\left(\frac{p}{q}\right)'$$

It remains to extend the derivative to negative rational numbers using the rule $q' = -(-q)'$. As one can freely move minus sign, it is easy to check that Leibniz rule holds also for products wher one or two factors are negative.

Remark. Easy to see that the derivative can be calculated by the same formula as in Problem 1:

$$n = p_1^{\alpha_1}p_2^{\alpha_2}\cdot\ldots\cdot p_k^{\alpha_k} \Rightarrow n' = n\sum_{i=1}^{k}\frac{\alpha_i}{p_i},$$

though here exponents can be also negative inegers.