How to derive a number?

Solutions of problems 7, 12 – 14, 15ce, 16.

Problem 7. Assume there is a constant M such that $i(n) < M$ for each n. Take all primes less or equal to n. We denote the number of such primes by $\pi(n)$. Then the number of pairs of such primes (including those of equal primes) is $\frac{\pi(n)(\pi(n)+1)}{2} \geqslant \frac{\pi(n)^2}{2}$. Each pair (p, q) with the sum $s \leqslant 2n$ give us a solution of the equation $x' = s$: $(pq)' = p + q = s$. So the number of pairs with a fixed sum s is not more then M. As there are less then $2n$ possible value for a sum, the total number of pairs is less then $2nM$. The inequality $2nM > \frac{\pi(n)^2}{2}$ implies $\pi(n) < 2\sqrt{Mn}$, what contradicts the well-known estimation $\pi(n) \geqslant C\frac{n}{\log n}$.

Problem 12. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ and $n' = n \sum \frac{\alpha_i}{p_i} = 0$. That implies

$$\alpha_1 p_2 \cdot \ldots \cdot p_k + \alpha_2 p_1 p_3 \cdot \ldots \cdot p_k + \ldots + \alpha_k p_1 p_2 \cdot \ldots \cdot p_{k-1} = 0.$$

Each term except for the ith is divisible by p_i, hence α_i is also divisible by p_i. Let us denote $\alpha_i = \beta_i p_i$ and rewrite the equation to the form $p_1 p_2 \cdot \ldots \cdot p_k (\beta_1 + \beta_2 + \ldots + \beta_k) = 0$. Now one can see that to get a zero derivative one must have $n = p_1^{\beta_1 p_1} p_2^{\beta_2 p_2} \cdot \ldots \cdot p_k^{\beta_k p_k}$, where $\sum \beta_i = 0$. It's easy to see that every such a number is a solution.

Problem 13. Let $x' = \alpha x$, where $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$. Then $x' = x \displaystyle\sum_{i=1}^{k} \frac{\alpha_i}{p_i}$ and $\alpha = \displaystyle\sum_{i=1}^{k} \frac{\alpha_i}{p_i} = \frac{a}{p_1 p_2 \cdot \ldots \cdot p_k}$, where a is an integer. So if the equation $x' = \alpha x$ has a non-trivial solution for $\alpha = \frac{a}{b}$, then all primes must be in the 1st power in the factorisation of b. Let us show that this condition is suffucient. We can point out a rational solution x for each $\alpha = \frac{a}{b}$, where $b = p_1 p_2 \cdot \ldots \cdot p_k$. Let us represent our fraction $\frac{a}{p_1 p_2 \cdot \ldots \cdot p_k}$ as a sum of numbers $\frac{\alpha_i}{p_i}$. Then the number $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ satisfies the equation $x' = \alpha x$.

In fact, such a representation is called a decomposition into the elementary fractions. We can build it by induction by the number of primes. If we have the only prime factor then the fraction itself is the decomposition. Let us show this fact for $\frac{a}{p_1 p_2 \cdot \ldots \cdot p_{k+1}}$. As $(p_1 p_2 \cdot \ldots \cdot p_k, p_{k+1}) = 1$, one can find with the Euclidian algorithm integers c и d, such that $c\,(p_1 p_2 \cdot \ldots \cdot p_k) + d p_{k+1} = 1$. This gives us $\frac{c}{p_{k+1}} + \frac{d}{p_1 p_2 \cdot \ldots \cdot p_k} = \frac{1}{p_1 p_2 \cdot \ldots \cdot p_{k+1}}$. By induction we can decompose both terms in the left part. If we multiply both sides with a we will get the necessary decomposition.

Let us describe all solutions of the equation $x' = \alpha x$. Let a and b be any two solutions (as we have shown there exist at least one solution). Then $\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2} = \frac{\alpha ab - a\alpha b}{b^2} = 0$; this means that $\frac{a}{b}$ satisfies the equation $y' = 0$. Vice versa, if y satisfies the equation $y' = 0$, and x_0 satisfies the equation $x' = \alpha x$, then $(x_0 y)' = \alpha x_0 y + 0 \cdot x_0$ и $x_0 y$ also satisfies the equation $x' = \alpha x$. So all the solutions of the equation $x' = \alpha x$ can be represented as $x_0 y$, where x_0 is a fixed solution of $x' = \alpha x$, and y is any solution of $y' = 0$.

Problem 14. Answer: not necessary. It is easy to build a counterexample using only elementary means.

Each prime p lies between two consequent powers of two. Increasing p one can make their exponents sufficiently large. Let us fix some positive integer n and choose such a p that makes the exponents greater then n. Then we can write $2^{k+n} < p < 2^{k+n+1}$, where

k is a positive integer. Then $\frac{1}{2^{n+1}} < \frac{2^k}{p} < \frac{1}{2^n}$, which implies that the sequence of fractions $\frac{2^k}{p}$ converges to 0. Let us derivate the fractions: $(\frac{2^k}{p})' = \frac{2^k}{p}(\frac{k}{2} - \frac{1}{p})$. When p increases the term $\frac{1}{p}$ tends to 0, but the term $\frac{k}{2}$ tends to infinity. So one can make $\frac{k}{2} - \frac{1}{p} > 2^{n+1}$. That implies $(\frac{2^k}{p})' > 1$, which means that the sequence of the derivatives does not tend to 0.

(For pedants: for each n we choose the least prime that satisfies $p_n > 2^{2^{n+2}+n+1}$, then denote k_n the greatest integer which satisfies $p_n > 2^{k_n+n+1}$ and take $x_n = \dfrac{2^{k_n}}{p_n}$. Check yourself that all the inequalities we have stated are valid!)

Problem 15c.

$$\left(-\frac{1}{16}\right)' = \frac{1}{8}$$

$$\left(-\frac{21}{16}\right)' = -\frac{16 \cdot 10 - 21 \cdot 32}{16^2} = 2; \quad \left(-\frac{43}{64}\right)' = -\frac{64 \cdot 1 - 43 \cdot 192}{64^2} = 2$$

Problem 15e. Yes, it can. One can take any $a \neq 0$ such that $a' = 0$. Then for any prime p the derivative $(pa)' = 1 \cdot a + pa' = a$, that gives us infinitely many solutions of the equation $x' = a$.

Problem 16a. It will be easy to show if we could manage to represent all the terms $x^{(k)}$ as fractions with the same denominator. So, let $x = \frac{a}{b}$, where $b = p_1 \cdot \ldots \cdot p_n$ (here p_i are primes, not necessary different). Then one can take as the common denominator the number $M = p_1^{p_1} \ldots p_n^{p_n}$. As $b|M$ we have $x = \frac{C_0}{M}$, where C_0 is an integer. Let us show that $x' = \frac{C_1}{M}$, where C_1 is an integer. In fact, $M' = nM$, so $x' = \left(\frac{C_0}{M}\right)' = \frac{C_0' M - C_0 M'}{M^2} = \frac{C_0' - C_0 n}{M}$, and the number $C_0' - C_0 n$ is evidently an integer. Induction by k shows that also $x^{(k)} = \frac{C_k}{M}$, where C_k is an integer.

Now we multiply the entire sequence with M. The new sequence still converges. But it consists of integers! That implies that from some k_0 all terms C_k are equal to each other, and the same is valid for the original sequence.