How to derive a number?

Suggested by V.Ufnarovski, presented by A.Shapovalov and A.Chebotarev

We define a derivative for positive integers as follows:

Definition. $p' = 1$, if p is prime;

$(mn)' = mn' + nm'$.

This definition implies $1' = 0$; let's also hold that $0' = 0$. Thus, derivative is a mapping of the set of nonnegative integers to itself.

Problem 1. a) Prove that the derivative of any integer is uniquely defined ;

b) find an explicit formula for the derivative.

Problem 2. Prove that

a) if $(a+b)' \leqslant a' + b'$ then inequality $(ka+kb)' \leqslant (ka)' + (kb)'$ holds for any k;

b) if $(a+b)' \geqslant a' + b'$ then inequality $(ka+kb)' \geqslant (ka)' + (kb)'$ holds for any k.

Problem 3. Prove following statements for any k:

a) $(3k)' = k' + (2k)'$;

b) $(2k)' \geqslant 2k'$;

c) $(5k)' = (3k)' + (2k)'$.

Problem 4. Solve the equation $n' = a$ for each $a \leqslant 10$.

Problem 5. a) Prove that $n' \leqslant n^2$;

b) Find a more precise upper estimation for n';

c) Find a lower estimation for n', where n is a composite.

Problem 6. Solve the equation $n' = n$.

Problem 7. Let $i(a)$ be a number of integers n such that $n' = a$. Prove that for each M there exists a such that $i(a) > M$.

Open problem 8. Is it true that equation $n'' = 1$ has infinitely many solutions?

Open problem 9. Is it true that equation $n' = 2b$ has a solution for any integer $b > 1$?

Open problem 10. Does there exist n such that $n' \neq n$ and

a) $n'' = n$?

b) $n^{(k)} = n$?

Problem 11. Generalize our definition to rational numbers. Prove that such a generalization is unique.