How to derive a number?

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Second part changed August 6.

New 15d, 15e. Modified 13a, 13b, 15a, 15b. Corrected 15b, 15c, 16a.

We extend our derivative to rational numbers as follows:

Definition. $(-x)' = -x'$, $\left(\dfrac{m}{n}\right)' = \dfrac{nm' - mn'}{n^2}$.

Problem 12. Solve the equation $x' = 0$ in rationals.

Problem 13. a) Find all rationals α, such that the equation $x' = \alpha x$ has a rational solution $x \neq 0$;

b) Prove that if the equation $x' = \alpha x$ has a nonzero solution then it has infinitely many rational solutions.

c) Describe all rational solutions of the equation $x' = \alpha x$.

Problem 14. A sequence of rational numbers $\{x_n\}$ tends to zero. $\{x'_n\}$ stands for the sequence of derivatives. Is it true that $\{x'_n\}$ also tends to zero?

Problem 15. a) Let A be the set of rationals a such that the equation $x' = a$ has a solution. Describe $A \cap \mathbb{N}$ as completely as possible.

b)* Does A coinside with the set of all rationals?

c) Do rationals $a = 2$ or $a = \frac{1}{8}$ belong to A?

d)* Does the equation $x' = 1$ has any positive rational solutions except for the primes?

e)* Can an equation $x' = a$ has infitely many rational solutions for any a except for $0, 1, -1$.

Problem 16. a) A sequence $\{x^{(k)}\} = x', x'', x''', \ldots, x^{(k)}, \ldots$ converges. Prove that all but a finite number of their terms are equal.

b)* Can a sequence of consequent derivatives $\{x^{(k)}\}$ be periodical with a period at least 3?

Problem 17. a)* Is it true that for every N there exists an integer $m > N$ with a derivative of some order $m^{(k)} = 0$?

b)* Does there exist an infinite increasing sequence of positive integers $\{n_i\}$, such that $n'_{i+1} = n_i$ for each i?

* Here an asterisk marks open problems.